Lecture 7

Transportation Problem (TP)

special case of Integer Linear Programming

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What is the Transportation Problem?

The transportation problem is a special type of linear programming problem where the objective is to minimize the cost of distributing a product from a number of sources or origins to a number of destinations.

Transportation Example

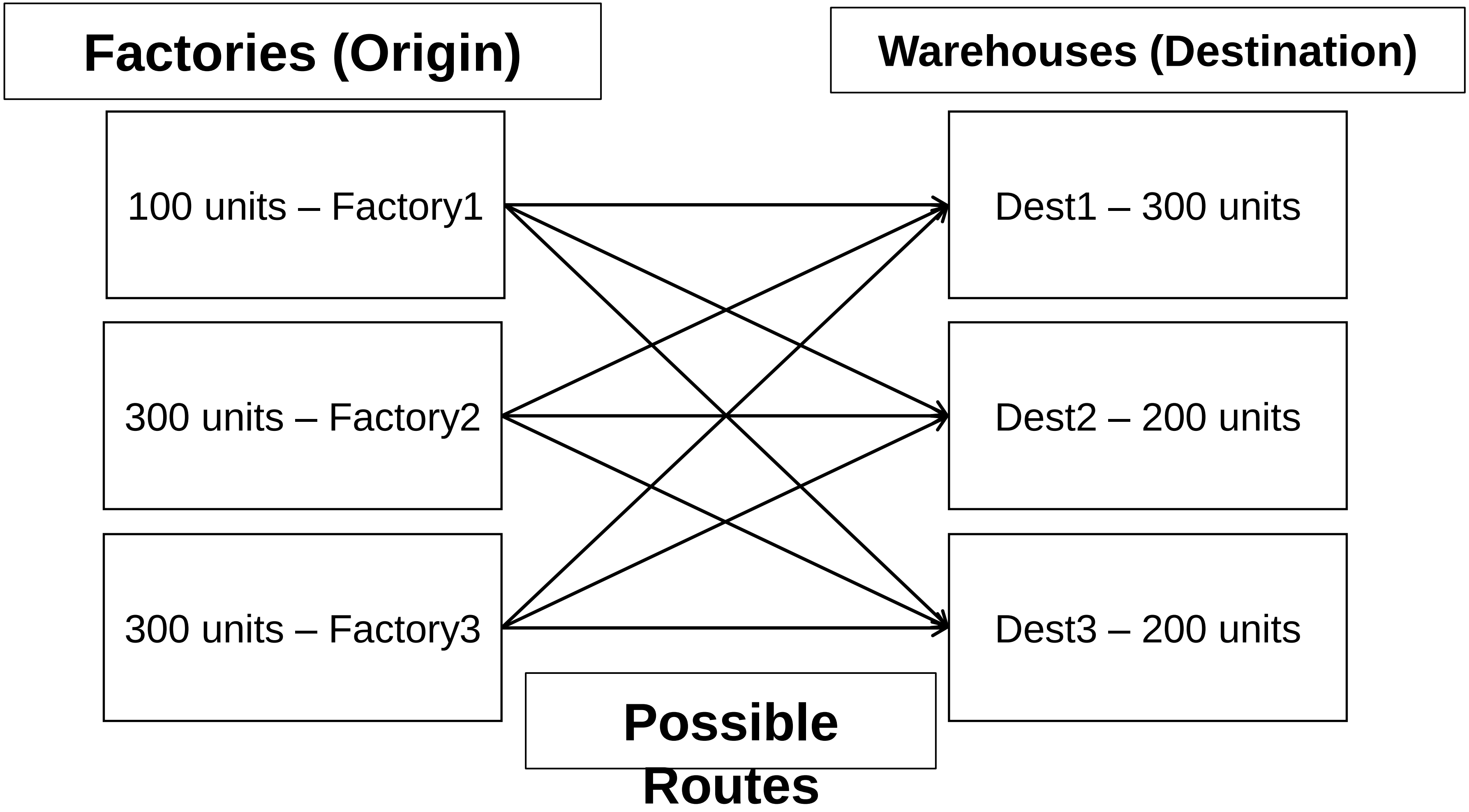
We have 3 factories and 3 warehouses

Decision: How much to ship from each origin to each destination?

Objective: Minimize transportation cost.

Transportation Cost per Unit			
Factory	Dest 1 (300 units)	Dest2 (200 units)	Dest3 (200 units)
Factory1(100 units)	Rs. 5	Rs. 4	Rs. 3
Factory2 (300 units)	Rs. 8	Rs. 4	Rs. 3
Factory3 (300 units)	Rs. 9	Rs. 7	Rs. 5

Transportation Example



Factories (Origin)

Warehouses (Destination)

100 units – Factory1

300 units – Factory2

300 units – Factory3

Dest1 – 300 units

Dest2 – 200 units

Dest3 – 200 units

Possible Routes

Decision Variables:

x_{ij} = number of units transported from factory “i” to Destination “j”

Transportation Cost per Unit			
Factory	Dest1 (300 units)	Dest2 (200 units)	Dest3 (200 units)
Factory1 (100 units)	x_{11}=? C_{11}=Rs. 5	x_{12} =? C_{12}=Rs. 4	x_{13}=? C_{13}=Rs. 3
Factory2 (300 units)	Rs. 8	Rs. 4	Rs. 3
Factory3 (300 units)	Rs. 9	Rs. 7	Rs. 5

Objective Function:

Min: Z= $5x_{11} + 4\,x_{12} + 3\,x_{13}$
$+ 8\,x_{21} + 4\,x_{22} + 3\,x_{23}$
$+ 9\,x_{31} + 7\,x_{32} + 5\,x_{33}$

supply Constraints:

$$x_{11} + x_{12} + x_{13} \leq 100$$
$$x_{21} + x_{22} + x_{23} \leq 300$$
$$x_{31} + x_{32} + x_{33} \leq 300$$

demand Constraints:

$$x_{11} + x_{21} + x_{31} \geq 300$$
$$x_{12} + x_{22} + x_{32} \geq 200$$
$$x_{13} + x_{23} + x_{33} \geq 200$$

The Transportation Algorithm

Unbalanced Problems

- If **(Total Supply) > (Total Demand)**,

- then for Excess supply is assumed to go to the inventory and costs nothing for shipping.
- Dummy destination column is added, **whose demand equals the difference between the total supply and total demand** and zero transportation cost

- If **(Total Supply) < (Total Demand)**,

- **a *dummy source* is created, whose supply equals the difference.**
- **All unit shipping costs into a dummy destination or out of a dummy source are 0.**

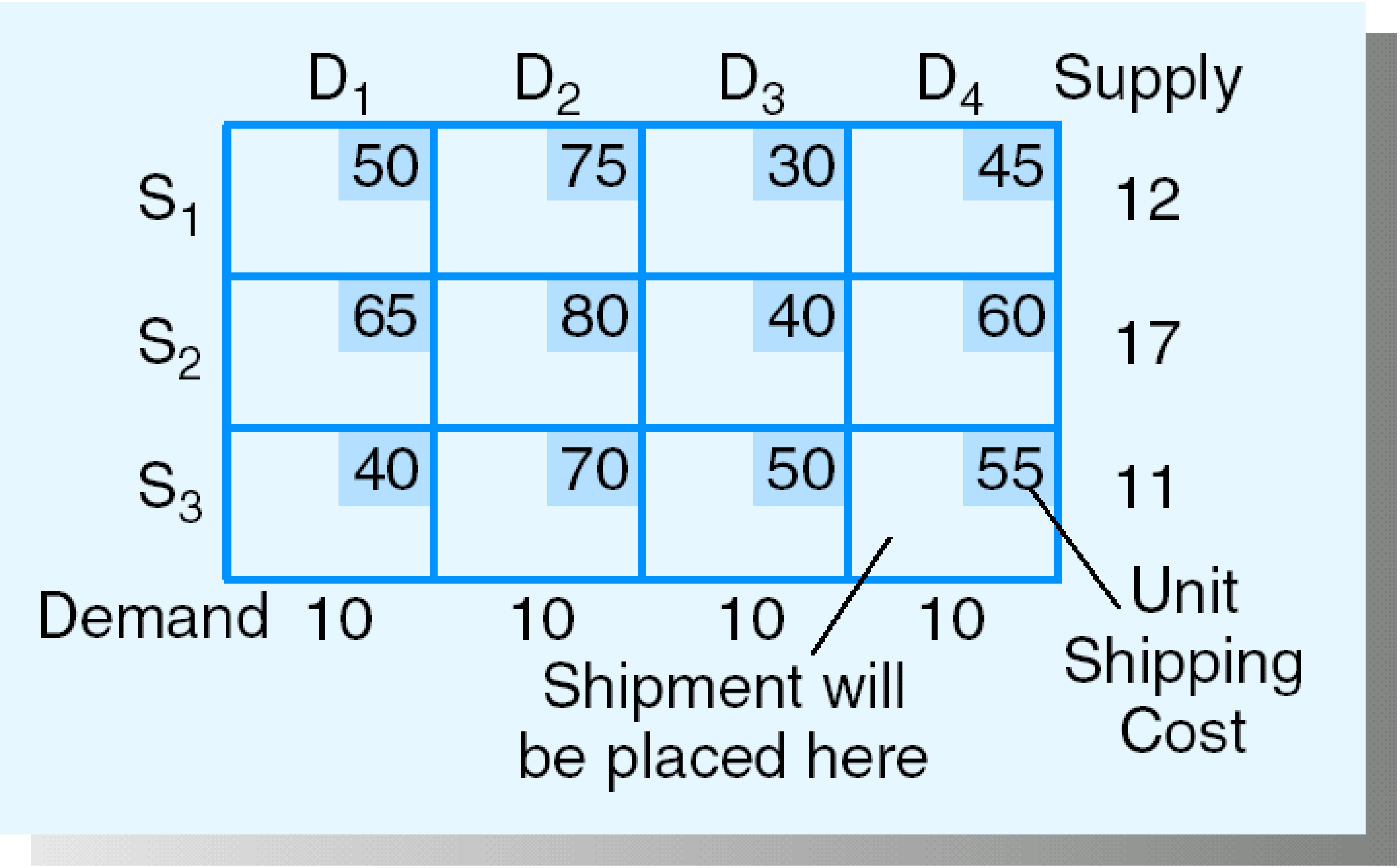
Example 1: Balanced Problem

		DESTINATIONS				Supply
		D_1	D_2	D_3	D_4	
	S_1	50	75	30	45	12
Sources	S_2	65	80	40	60	17
	S_3	40	70	50	55	11
Demand		10	10	10	10	

Example 2: Unbalanced Problem

	Destination				**Supply**
	D1	**D2**	**D3**	**D4**	
S1	50	75	35	75	12
S2	65	80	60	65	17
S3	40	70	45	55	11
(Dummy)	**0**	**0**	**0**	**0**	**10**
Demand	15	10	15	10	

Transportation Tableau:



	D_1	D_2	D_3	D_4	Supply
S_1	50	75	30	45	12
S_2	65	80	40	60	17
S_3	40	70	50	55	11
Demand	10	10	10	10	

Shipment will
be placed here

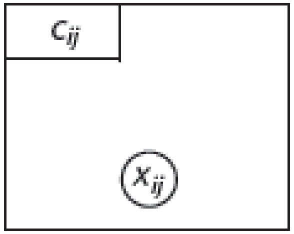
Unit
Shipping
Cost

TABLE 9.15 Format of a transportation simplex tableau

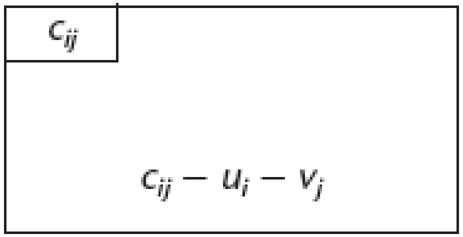
		Destination				Supply	u_i
		1	**2**	⋯	***n***		
Source	1	c_{11} x_1	c_{12} x_1	⋯	c_{1n} x_1	s_1	
	2	c_{21} x_2	c_{22} x_2	⋯	c_{2n}	s_2	
	⋮	⋯	⋯	⋯	⋯	⋮	
	m	c_{m1} x_n	c_{m2}	⋯	c_{mn}	s_m	
Demand		d_1	d_2	⋯	d_n	$Z =$	
	v_j						

Additional information to be added to each cell:

If x_{ij} is a basic variable



If x_{ij} is a nonbasic variable



The Transportation Problem

- Transportation method is more efficient
  - Especially for large problems
- For transportation problems with *m* sources and n destinations:
  - Number of basic variables is equal to $\boldsymbol{m+n-1}$

The Transportation Method

The method involves **3 steps**:

1. **Obtaining Initial Basic Feasible Solution**
   1. North-West Corner Rule
   2. Vogel's Approximation Method
2. **Testing the Optimality**

3. **Improving the Solution**

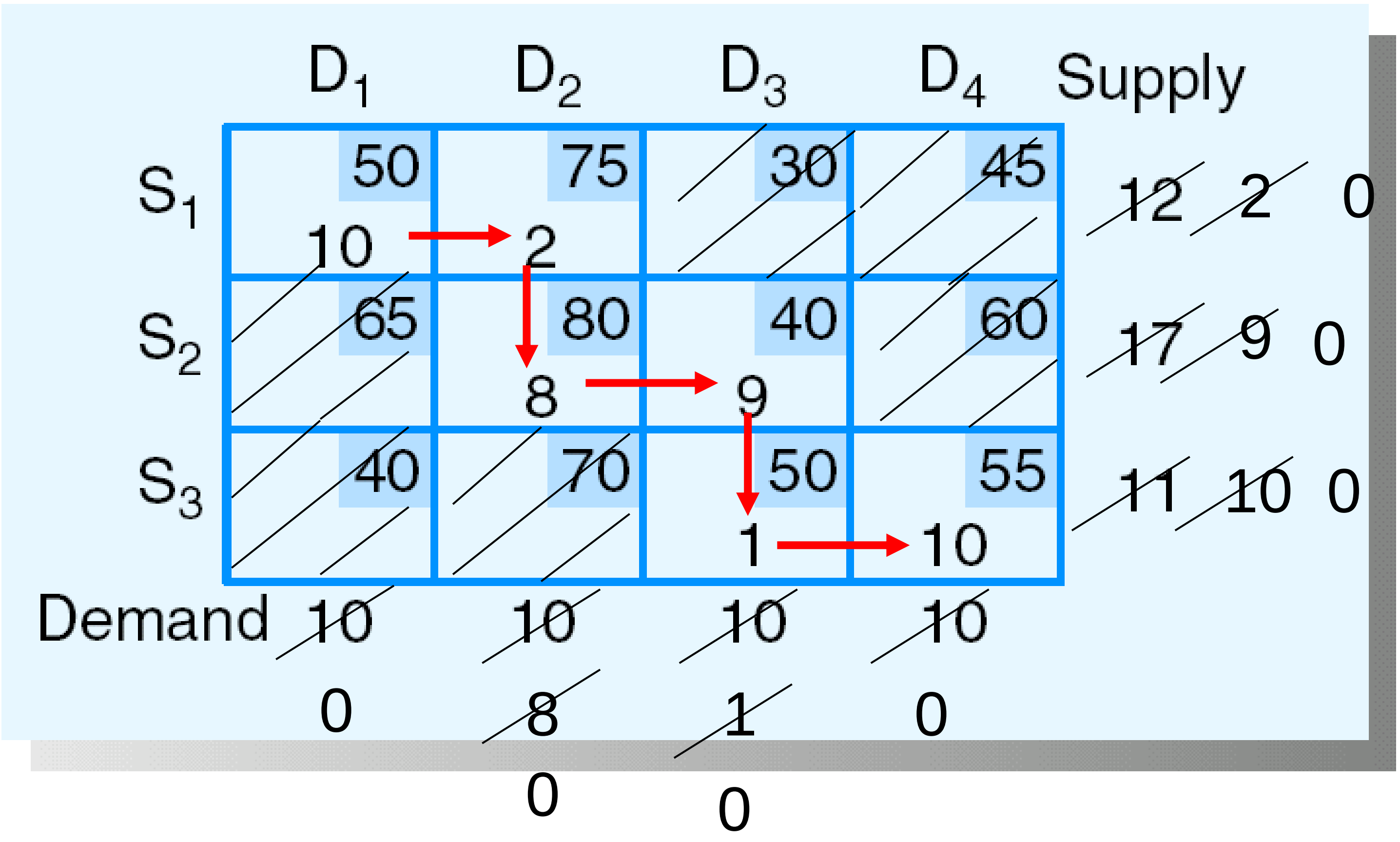
1- Initial Solution Procedure:

1. Northwest Corner Starting Procedure

1. Select the variable in the upper left (northwest) corner
2. Allocate the minimum of s or d to this variable. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate all variables in the column from future consideration and reduce the supply in its row by d.

REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

Example 1:



	D_1	D_2	D_3	D_4	Supply
S_1	50 10	75 2	30	45	~~12~~ ~~2~~ 0
S_2	65	80 8	40 9	60	~~17~~ ~~9~~ 0
S_3	40	70	50 1	55 10	~~11~~ ~~10~~ 0
Demand	~~10~~ 0	~~10~~ ~~8~~ 0	~~10~~ ~~1~~ 0	~~10~~ 0	

Total shipping cost = 2250

Example 2: Solve the Transportation Table to find Initial Basic Feasible Solution using North-West Corner Method.

Total Cost =19*5+30*2+30*6+40*3+70*4+20*14

= Rs. 1015

	D_1	D_2	D_3	D_4	Supply
S_1	19 5	30 2	50	10	7
S_2	70	30 6	40 3	60	9
S_3	40	8	70 4	20 14	18
Demand	5	8	7	14	34

Limitations of North West Corner Rule

- Although this method is relatively simple, it is not efficient in terms of cost minimizing. Because it takes into account only the available supply and demand requirements in making assignments and takes no account of the transportation cost involved.

The Transportation Method

Broadly the method involves 3 steps:

1. **Obtaining Initial Basic Feasible Solution**
   - a. North-West Corner Rule √
   - b. Vogel’s Approximation Method
2. **Testing the Optimality**

3. **Improving the Solution**

3. Vogel's Approximation Method Starting Procedure

1. For each remaining row and column, determine the difference between the lowest two *remaining* costs; these are called the *row and column penalties*.

2. Select the row or column with the largest difference found in step 1 and note the supply remaining for its row, s, and the demand remaining in its column, d.

3. Allocate the minimum of **s** or **d** to the variable in the selected row or column with the lowest remaining unit cost. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate all variables in the column from future consideration and reduce the supply in its row by d.

REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

Example 1

	D_1	D_2	D_3	D_4	Supply	
S_1	50	75	30	45	12	15
S_2	65	80	40 10	60	~~17~~ 7	20*
S_3	40	70	50	55	11	10
Demand	10	10	10	10		
	10	5	10	10		

Max Diff

Min cost

Min (10, 17)

Example 1 cont.

	D_1	D_2	D_3	D_4	Supply
S_1	50	75	30	45	12
S_2	65	80	40 10	60	~~17~~ 7
S_3	40 10	70	50	55	~~11~~ 1
Demand	10	10	10	10	

15	5	
20**	5	
10	**15****	

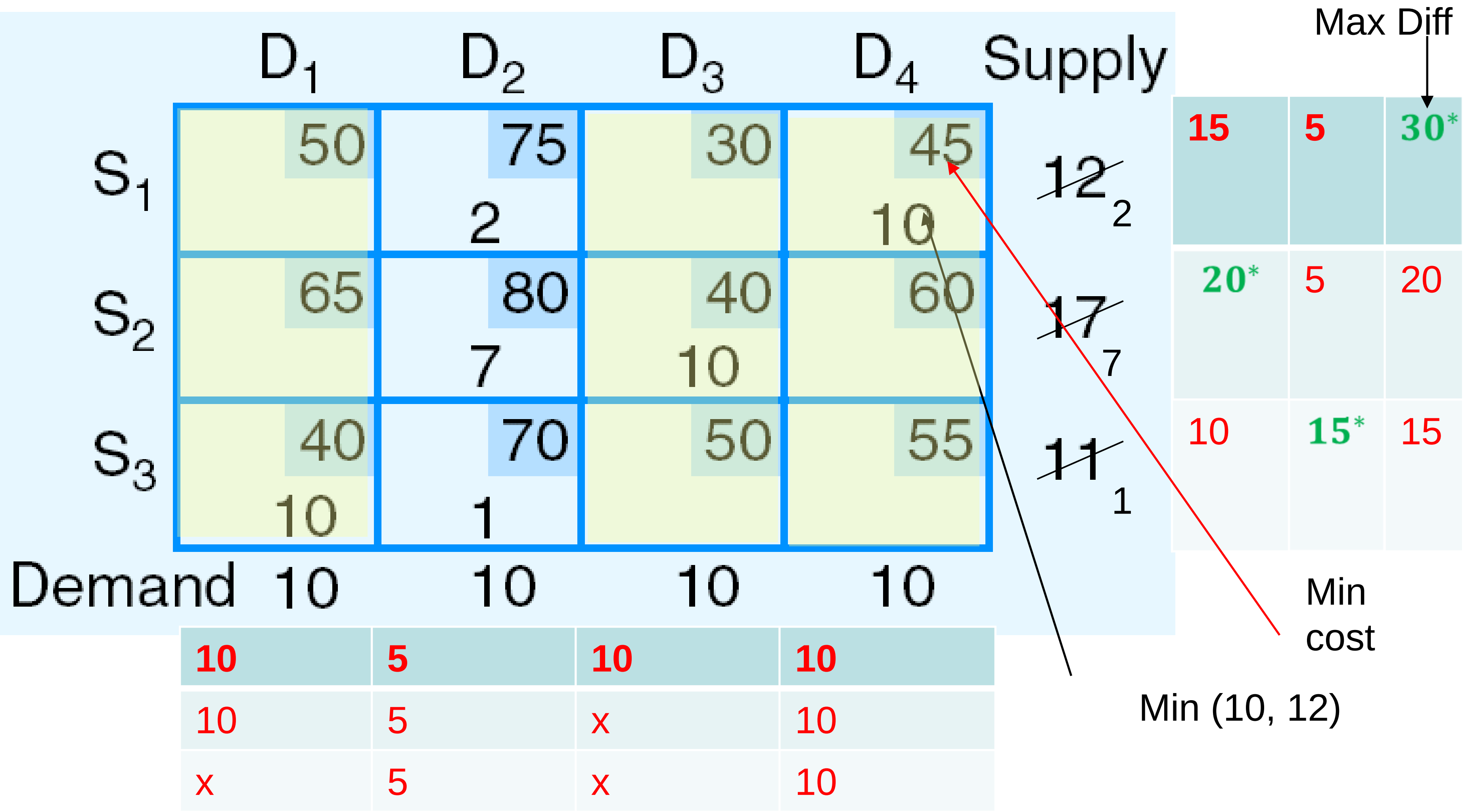
Max Diff

10	5	10	10
10	5	x	10

Min (10, 11)

Min
cost

Example 1 cont.



	D1	D2	D3	D4	Supply
S1	50	75 2	30	45 10	~~12~~ 2
S2	65	80 7	40 10	60	~~17~~ 7
S3	40 10	70 1	50	55	~~11~~ 1
Demand	10	10	10	10	

Max Diff

15	**5**	**30***
20*	5	20
10	15*	15

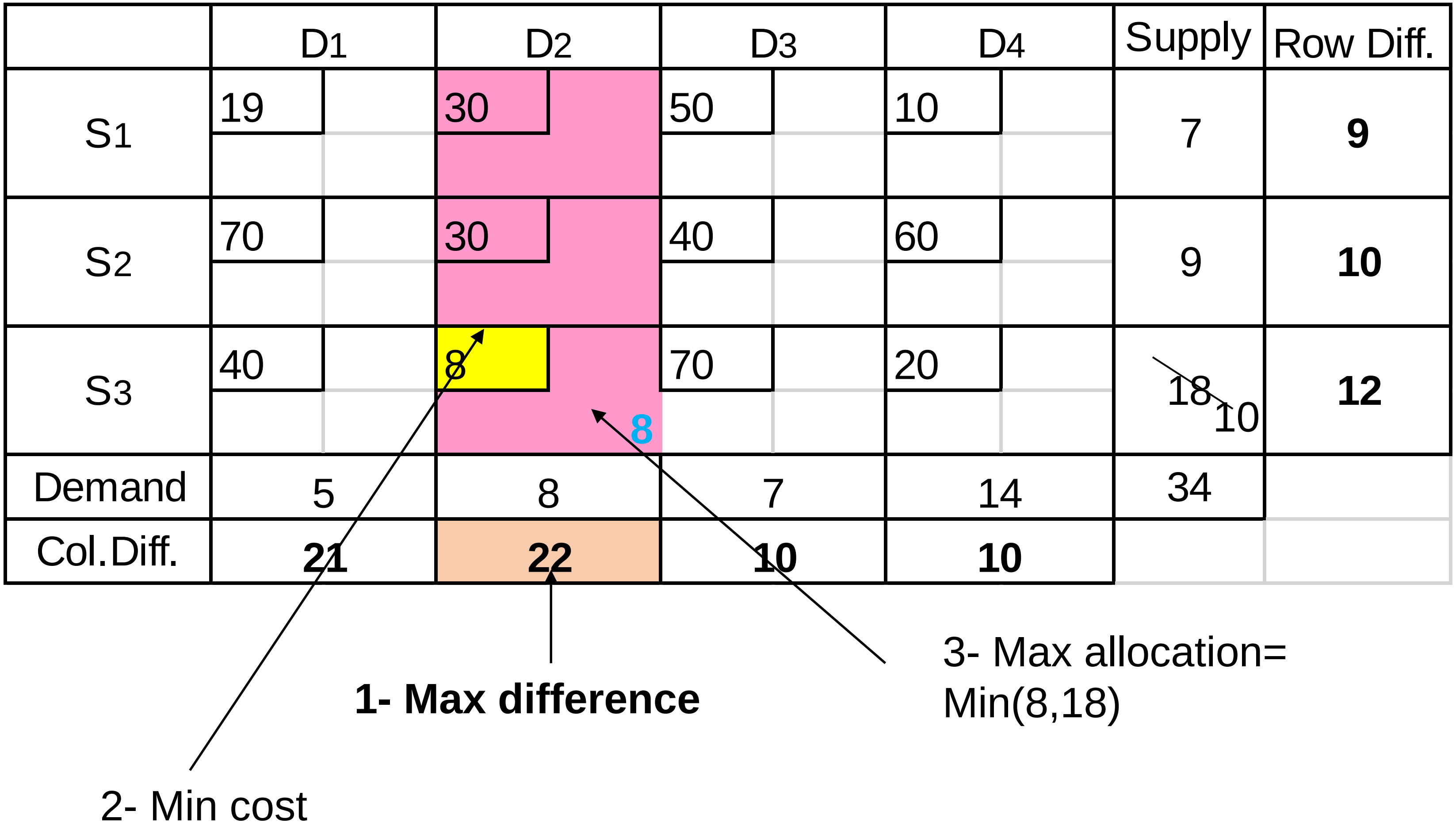
10	**5**	**10**	**10**
10	5	x	10
x	5	x	10

Min cost

Min (10, 12)

Total sipping cost = 2030 while in NW Total sipping cost = 2250

Example 2:



	D1	D2	D3	D4	Supply	Row Diff.
S1	19	30	50	10	7	**9**
S2	70	30	40	60	9	**10**
S3	40	8 8	70	20	~~18~~ 10	**12**
Demand	5	8	7	14	34	
Col.Diff.	**21**	**22**	**10**	**10**		

1- Max difference

2- Min cost

3- Max allocation= Min(8,18)

Example 2:

	D1	D3	D4	Supply	Row Diff.
S1	19 **5**	50	10	7	**9**
S2	70	40	60	9	**20**
S3	40	70	20	10	**20**
Demand	5	7	14	34	
Col.Diff.	**21**	**10**	**10**		

Example 2:

	D3	D4	Supply	Row Diff.
S1	50	10	2	**40**
S2	40	60	9	**20**
S3	70	20 **10**	10	**50**
Demand	7	14	34	
Col.Diff.	**10**	**10**		

Example 2:

	D3	D4	Supply	Row Diff.
S1	50	10 **2**	2	**40**
S2	40	60	9	**20**
Demand	7	4	34	
Col.Diff.	**10**	**50**		

Example 2:

	D3	D4	Supply	Row Diff.
S2	40 **7**	60 **2**	9	**20**
Demand	7	2	34	
Col.Diff.				

Example 2:

	D1	D2	D3	D4	Supply	
S1	19 5	30	50	10 2	7	
S2	70	30	40 7	60 2	9	
S3	40	8 8	70	20 10	18	
Demand	5	8	7	14	34	

Example 2:

The total transportation cost obtained by this method
= 8*8+19*5+20*10+10*2+40*7+60*2
= Rs.779

while by using Northwest
Total Cost =19*5+30*2+30*6+40*3+70*4+20*14
= Rs. 1015

Here, we can see that ***Vogel's Approximation Method*** involves the lowest cost than *North-West Corner Method* hence is the most preferred method of finding initial basic feasible solution.

The Transportation Method

Broadly the method involves 3 steps:

1. **Obtaining Initial Basic Feasible Solution**
    a. North-West Corner Rule √
    b. Vogel's Approximation Method √
2. **Testing the Optimality**

3. **Improving the Solution**

2. Testing the Optimality

Find an initial basic feasible solution by some starting procedure. Then,

1. Set $U_i = 0$ or $V_j = 0$ the row or column with the largest number of basic variables. Solve for the other U_i's and V_j's by:

$$C_{ij} = U_i + V_j$$ **for basic variables.**

Then calculate the C_{ij}–Z_{ij} values for non-basic variables by:

$$C_{ij} - Z_{ij} = C_{ij} - U_i - V_j$$

A Basic Feasible Solution is Optimal if $C_{ij} - U_i - V_j \geq 0$ for each (i, j)

- If all C_{ij}–Z_{ij} values are nonnegative, STOP; the current solution is optimal.

Else, the current solution is not optimal then,

Improve the solution

1- Choose the non-basic variable with the most negative $C_{ij} - U_i - V_j$ value as the entering variable

2. Find the cycle that includes the entering variable and some of the BASIC variables.

Cycle Properties:

1- Begin with the Most negative $C_{ij} - U_i - V_j$ (Entering Variable)
2- All the elements of the loop are basic Variable except the beginning one (E.V.).

3- The number of the elements in the Loop even (min 4 elements)

4- There is no consecutive 3 elements in the same row or the same column.

5- move to left or right or top or bottom but not Diagonal

-→ There is a Unique Loop in each basic feasible Solution

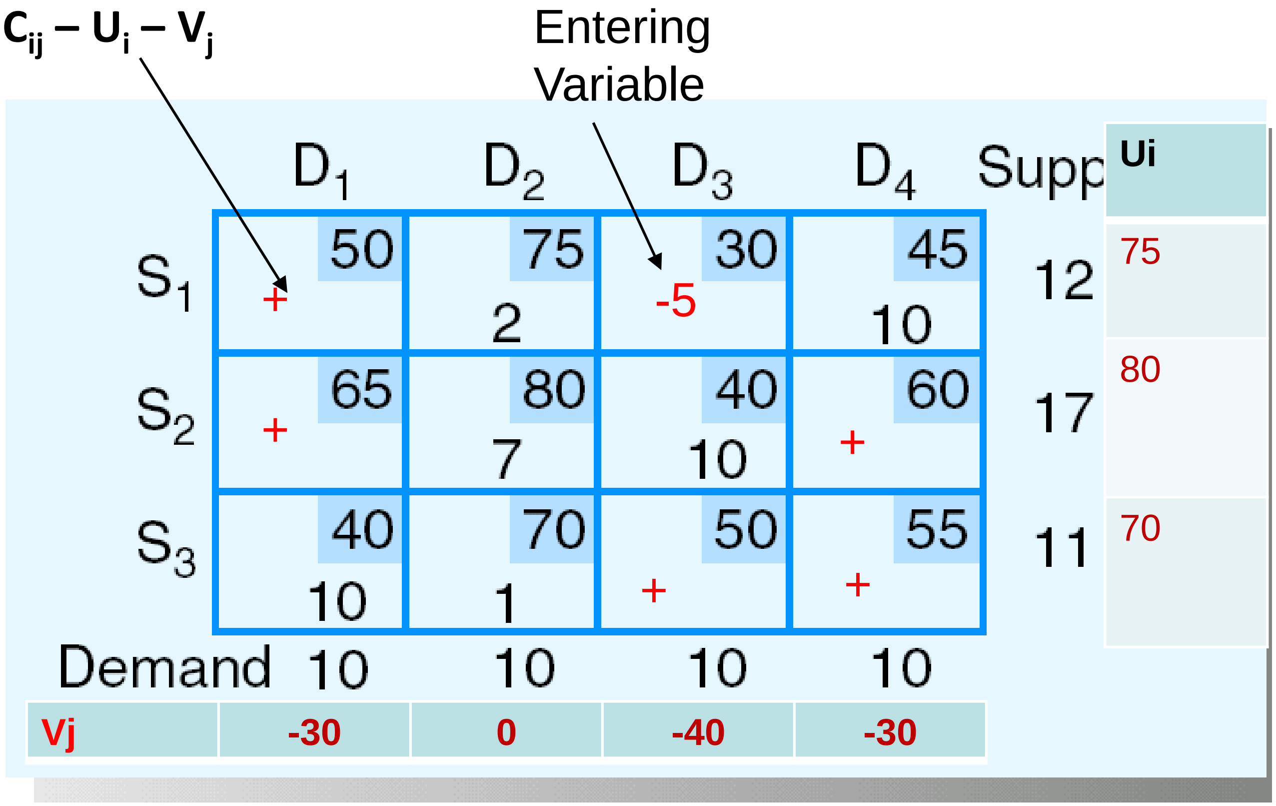
Note:

There must be m + n - 1 basic variables for the transportation simplex method to work!

=> Add dummy source or dummy destination, if necessary

(m=# of sources and n=# of destinations)

Example 1:



$C_{ij} – U_i – V_j$

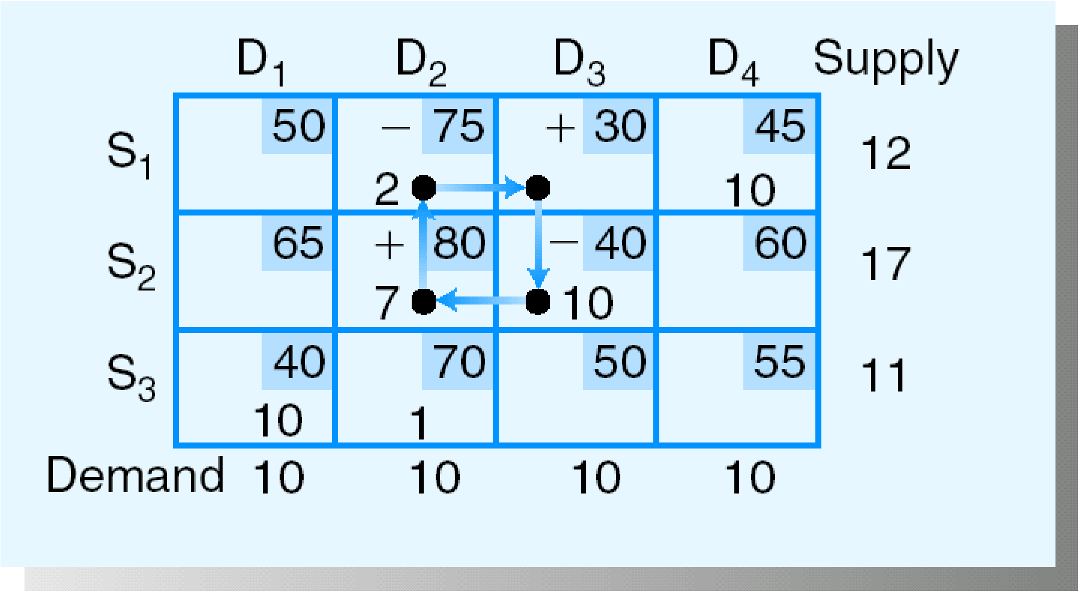
Entering Variable

	D_1	D_2	D_3	D_4	Supp	**Ui**
S_1	50 +	75 2	30 -5	45 10	12	75
S_2	65 +	80 7	40 10	60 +	17	80
S_3	40 10	70 1	50 +	55 +	11	70
Demand	10	10	10	10		
Vj	**-30**	**0**	**-40**	**-30**		

Total sipping cost = 2030

Example 1:

Step 2: Determine Which Current Basic Variable Reaches 0 First



	D_1	D_2	D_3	D_4	Supply
S_1	50	− 75 2	+ 30	45 10	12
S_2	65	+ 80 7	− 40 10	60	17
S_3	40 10	70 1	50	55	11
Demand	10	10	10	10	

Note: 1. Cycle property

2. X_{12} is the leaving variable

Entering Variable

Donor
2-2

Recipient
2

Recipient
7+2

Donor
10-2

All Recipient + Min Donor
All Donor – Min Donor

Min Donor=2

Example 1:

Step 3: Determine the Next Transportation Tableau

	D_1	D_2	D_3	D_4	Supply	**Ui**
S_1	(+) 50	(+) 75	30 2	45 10	12	0
S_2	(+) 65	80 9	40 8	(+) 60	17	10
S_3	40 10	70 1	(+) 50	(+) 55	11	0
Demand	10	10	10	10		
Vj	**40**	**70**	**30**	**45**		

Total shipping cost = 2020

{Improvement = 2 (-5) = 10 }

Departing Variable

$$C_{ij} - U_i - V_j \geq 0 \text{ for each } (i,j) \rightarrow Optimal\ Soln.$$

Donor

X

Entering Variable Recipient

2

Recipient

9

Donor

8

Example 1:

Total shipping cost = 30*20+45*10+9*80+8*40+10*40+70*1

=2020

$\boldsymbol{Since\ C_{ij} - U_i - V_j \geq 0}$, **for each** $\boldsymbol{(i, j) \rightarrow Optimal\ Soln}$

Example 2:

	D1	D2	D3	D4	Supply	ui
S1	19 5	30 (+)	50 (+)	10 2	7	**cij-vj=10**
S2	70 (+)	30 E.V -18	40 7	60 2	9	**60**
S3	40 (+)	8 8	70 (+)	20 10	18	**20**
Demand	5	8	7	14	34	
vj	**9**	**-12**	**-20**	**0**		

Loop:

Recipient

Donor

2

Min Donor

Donor

8

Recipient

10

All Recipient + Min Donor

All Donor – Min Donor

Min Donor will be the Departing Variable

Recipient

+2

Donor

2-2

Donor

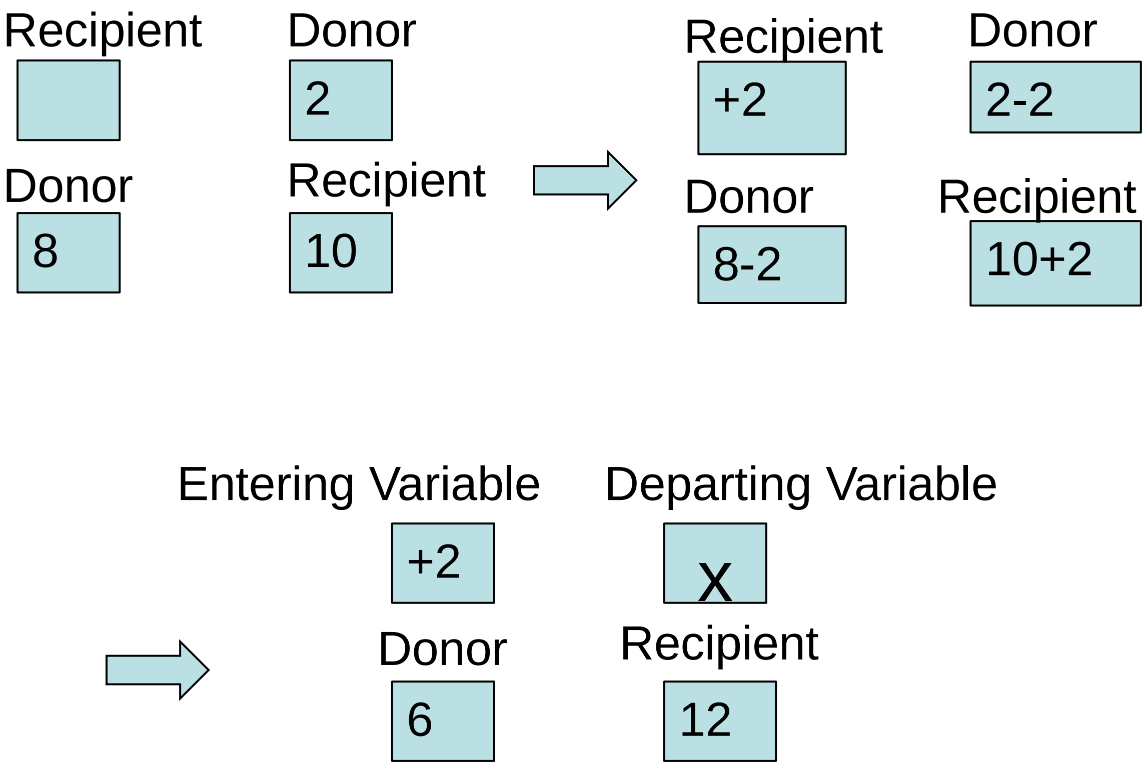
8-2

Recipient

10+2

Example 2:

All Recipient + Min Donor
All Donor – Min Donor



Recipient	Donor
	2
Donor	Recipient
8	10

→

Recipient	Donor
+2	2-2
Donor	Recipient
8-2	10+2

→

Entering Variable	Departing Variable
+2	X
Donor	Recipient
6	12

Example 2:

Entering Variable

+2

Departing Variable

X

Donor

6

Recipient

12

	D1	D2	D3	D4	Supply	ui
S1	19 5	30 (+)	50 (+)	10 2	7	**0**
S2	70 (+)	30 E.V **2**	40 7	60 (+)	9	**32**
S3	40 (+)	8 **6**	70 (+)	20 **12**	18	**10**
Demand	5	8	7	14	34	
vj	**19**	**-2**	**8**	**10**		

Example 2:

The total transportation cost before improvement
= 8*8+19*5+20*10+10*2+40*7+60*2
= Rs.779

The total transportation cost after improvement
=19*5+10*2+30*2+40*7+8*6+20*12
= Rs. 743