Operations Research

Lecture 5

Regular Simplex Method

Simplex Algorithm

The key solution concepts

- Solution Concept 1: the simplex method focuses on BFS solutions.

- Solution concept 2: the simplex method is an iterative algorithm

(a systematic solution procedure that keeps repeating a fixed series of steps, called, an iteration, until a desired result has been obtained) with the following structure:

Simplex Algorithm

Initialization: setup to start iterations, including finding an initial BFS solution

Optimality test: Is the current BFS solution optimal?

- if yes → stop
- if no ↓

Iteration: Perform an iteration to find a better BFS solution

Simplex algorithm

- Solution concept 3: whenever possible, the initialization of the simplex method chooses the **origin point** (all decision variables equal zero) to be the initial BFS solution.

- Solution concept 4: Each time the simplex method performs an iteration to move from the current BFS solution to a better one, it always chooses a BFS solution that is adjacent to the current one.

Simplex algorithm

- Solution concept 5: After the current BFS solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this BFS solution. Each of these edges leads to an adjacent BFS solution at the other end, but the simplex method doesn't even take the time to solve for the adjacent BFS solution. Instead it simply identifies the rate of improvement in Z that would be obtained by moving along the edge. And then chooses to move along the one with largest positive rate of improvement.

The simplex method in tabular form

- Steps:

1. Initialization:

a. transform all the constraints to equality by introducing slack, surplus, and artificial variables as follows:

Constraint type	Variable to be added
$\geq$	+ slack (s)
$\leq$	- Surplus (s) + artificial (A)
$=$	+ Artificial (A)

Simplex method in tabular form

2. Construct the initial simplex tableau

Basic variable	X_1	…	X_n	S_1	……	S_n	A_1	….	A_n	RHS
S	Coefficient of the constraints **A**									b_1
⋮										⋮
A										b_m
Z	$C_0^t A - C^t$ (Maximization) $C^t - C_0^t A$ (Minimization)									Z=$C_0^t b$ -Z=-$C_0^t b$

Simplex method in tabular form

2. Test for optimality:

Case 1: Maximization problem

the current BF solution is optimal if every coefficient in the $C_0^t A - C^t$ row is nonnegative

Case 2: Minimization problem

the current BF solution is optimal if every coefficient in the $C^t - C_0^t A$ row is nonnegative

Or We can Solve Min as Max Because ***Min Z=Max (-z)***

Simplex method in tabular form

3. Iteration

Step 1: determine the entering basic variable by selecting the variable (automatically a nonbasic variable) with the most negative value

last row ($C_0^t A - C^t$). Put a box around the column below this variable, and call it the "pivot column"

Simplex method in tabular form

- *Step 2:* Determine the leaving basic variable by applying the minimum ratio test as following:

1. Pick out each coefficient in the pivot column that is **strictly positive (> 0)**
2. Divide each of these coefficients into the right hand side entry for the same row
3. Identify the row that has the smallest of these **ratios**
4. The basic variable for that row is the leaving variable, so replace that variable by the entering variable in the basic variable column of the next simplex tableau. Put a box around this row and call it the "pivot row"

Simplex method in tabular form

- *Step 3:* Solve for the new BF solution by using elementary row operations –
  - ✓multiply or divide a row by a nonzero constant;
  - ✓add or subtract a multiple of one row to another row)

to construct a new simplex tableau, and then return to the optimality test. The specific elementary row operations are:

1. Divide the pivot row by the "pivot number" (the number in the intersection of the pivot row and pivot column)
2. For each other row that has a negative coefficient in the pivot column, add to this row the product of the absolute value of this coefficient and the new pivot row.
3. For each other row that has a positive coefficient in the pivot column, subtract from this row the product of the absolute value of this coefficient and the new pivot row.

Simplex method

- Example (All constraints are $\leq$)

Solve the following problem using the simplex method

- Maximize

$$Z = 3X_1 + 5X_2$$

Subject to

$$X_1 \leq 4$$

$$2X_2 \leq 12$$

$$3X_1 + 2X_2 \leq 18$$

$$X_1, X_2 \geq 0$$

Simplex method

1. Standard form

Maximize $Z = 3X_1 + 5X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$X_1 + S_1 = 4$$
$$2X_2 + S_2 = 12$$
$$3X_1 + 2X_2 + S_3 = 18$$
$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Initial Basic Feasible solution:

$S_1 = 4$
$S_2 = 12$
$S_3 = 18$

} Basic Variables

$X_1 = X_2 = 0$ Non-Basic Variables

The solution at the initial tableau is associated to the origin point at which all the decision variables are zero.

Initial tableau

2. Initial tableau

Entering variable

Basic variable	X_1 3	X_2 5	S_1 0	S_2 0	S_3 0	RHS	ratio
S_1 0	1	0	1	0	0	4	-
S_2 0	0	2	0	1	0	12	6
S_3 0	3	2	0	0	1	18	9
	-3	-5	0	0	0	0	

Initial Basic Variables

Leaving variable

Pivot column

Most negative

Pivot number

Pivot row

Min ratio

Optimality test

- By investigating the last row of the initial tableau, we find that there are some negative numbers. Therefore, the current solution is not optimal

First Iteration

- Step 1: Determine the entering variable by selecting the variable with the most negative in the last row.
- From the initial tableau, in the last row ,the coefficient of X_1 is -3 and the coefficient of X_2 is -5; therefore, the most negative is -5. consequently, X_2 is the entering variable.
- X_2 is surrounded by a box and it is called the pivot column

First Iteration

- Step 2: Determining the leaving variable by using the minimum ratio test as following:

Basic variable	Entering variable X_2 (1)	RHS (2)	Ratio (2)÷(1)
S_1	0	4	None
S_2 Leaving	**2**	**12**	**6 Smallest ratio**
S3	2	18	9

First Iteration

- Step 3: solving for the new BF solution by using the eliminatory row operations as following:

1. New pivot row = old pivot row ÷ pivot number

Basic variable	X_1 3	X_2 5	S_1 0	S_2 0	S_3 0	RHS
S_1 0		0				
X_2 5	0	1	0	1/2	0	6
S_3 0		0				
Z		0				

Note that X_2 becomes in the basic variables list instead of S_2

SO its coefficient in the Tableau must be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

First Iteration

For S_3

	3	2	0	0	1	18
-						
2	(0	1	0	1/2	0	6)
	3	0	0	-1	1	6

Basic variable	X_1 3	X_2 5	S_1 0	S_2 0	S_3 0	RHS
S_1 0	1	0	1	0	0	4
X_2 5	0	1	0	1/2	0	6
S_3 0	3	2	0	0	1	18

Basic variable	X_1 3	X_2 5	S_1 0	S_2 0	S_3 0	RHS
S_1 0	1	0	1	0	0	4
X_2 5	0	1	0	1/2	0	6
S_3 0	3	0	0	-1	1	6

Substitute this values in the table

Second Iteration

This solution is not optimal, since there is a negative numbers in the last row

Basic variable	X_1 3	X_2 5	S_1 0	S_2 0	S_3 0	RHS	ratio
S_1 0	1	0	1	0	0	4	4
X_2 5	0	1	0	1/2	0	6	-
S_3 0	3	0	0	-1	1	6	2
Z	-3	0	0	5/2	0	30	

The most negative value; therefore, X_1 is the entering variable

The smallest ratio is 6/3 =2; therefore, S_3 is the leaving variable

Second Iteration

- Apply the same rules we will obtain this solution:

Basic variable	X_1 3	X_2 5	S_1 0	S_2 0	S_3 0	RHS
S_1 0	0	0	1	1/3	-1/3	2
X_2 5	0	1	0	1/2	0	6
X1 3	1	0	0	-1/3	1/3	2
Z	0	0	0	3/2	1	**36**

This solution is optimal; since there is no negative solution in the last row: basic variables are $X_1 = 2$, $X_2 = 6$ and $S_1 = 2$; the non-basic variables are $S_2 = S_3 = 0$ then $Z = 36$

This Solution is **Unique** Since the $\boldsymbol{Number\ of\ Zeroes\ in\ the\ Last\ Row = The\ number\ of\ Basic\ Variables}$

Special cases of linear programming

- Infeasible solution
- Multiple solution (infinitely many solution)
- Unbounded solution
- Degenerated solution

Special cases

- In the final tableau, if one or more artificial variables (A_1, A_2, …) still basic and has a nonzero value, then the problem has an infeasible solution
- When determining the leaving variable of any tableau, if there is no positive ratio (all the entries in the pivot column are negative and zeroes), then the solution is unbounded.
- A Solution that has a basic variable with zero value is called a "degenerate solution".
- If there is a zero under one or more nonbasic variables in the last tableau (optimal solution tableau), then there is a multiple optimal solution.

Simplex method incase of Artificial variables "Big M method"

- Solve the following linear programming problem by using the simplex method:
- Min $Z = 2X_1 + 3X_2$

Subject to

$$\frac{1}{2} X_1 + \frac{1}{4} X_2 \leq 4$$

$$X_1 + 3X_2 \geq 20$$

$$X_1 + X_2 = 10$$

$$X_1, X_2 \geq 0$$

Big M method

- Solution

Step 1: standard form

$$\text{Min}: Z = 2X_1 + 3X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$$

Subject to

$$\tfrac{1}{2}X_1 + \tfrac{1}{4}X_2 + S_1 = 4$$

$$X_1 + 3X_2 - S_2 + A_1 = 20$$

$$X_1 + X_2 + A_2 = 10$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Where: M is a very large number

Big M method

- Step 2: Initial tableau

Basic variables	X_1 2	X_2 3	S_1 0	S_2 0	A_1 M	A_2 M	RHS
S_1 0	½	¼	1	0	0	0	4
A_1 M	1	3	0	-1	1	0	20
A_2 M	1	1	0	0	0	1	10
Z	2-2M	3-4M	0	M	0	0	

Note that one of the simplex rules is violated, which is the basic variables A_1, and A_2 have a non zero value in the z row; therefore, this violation must be corrected before proceeding in the simplex algorithm as follows.

Big M method

- The initial tableau will be:

Basic variables	X_1 2	X_2 3	S_1 0	S_2 0	A_1 M	A_2 M	RHS	Ratio
S_1 0	1/2	1/4	1	0	0	0	4	16
A_1 M	1	3	0	-1	1	0	20	6.6
A_2 M	1	1	0	0	0	1	10	10
Z	2-2M	3-4M	0	M	0	0		

Min Ratio

Pivot

Most negative

- Since there is a negative value in the last row, this solution is not optimal
- The entering variable is X_2 (it has the most negative value in the last row)
- The leaving variable is A_1 (it has the smallest ratio)

Big M method

- First iteration

Basic variables	X_1 2	X_2 3	S_1 0	S_2 0	A_1 M	A_2 M	RHS
S_1 0	5/12	0	1	1/12	-1/12	0	7/3
X_2 3	1/3	1	0	-1/3	1/3	0	20/3
A_2 M	2/3	0	0	1/3	-1/3	1	10/3
Z	1-2/3M	0	0	1-1/3M	4/3M-1	0	

Most negative

- Since there is a *negative* value in the last row, this solution is not optimal
- The entering variable is X_1 (it has the most *negative* value in the last row)
- The leaving variable is A_2 (it has the smallest ratio)

Big M method

- Second iteration

Basic variables	X_1 2	X_2 3	S_1 0	S_2 0	A_1 M	A_2 M	RHS
S_1 0	0	0	1	-1/8	1/8	-5/8	1/4
X_2 3	0	1	0	-1/2	1/2	-1/2	5
X_1 2	1	0	0	1/2	-1/2	3/2	5
Z	0	0	0	1/2	M-1/2	M-3/2	25

This solution is optimal, since there is no negative value in the last row. The optimal solution is:

$X_1 = 5$, $X_2 = 5$, $S_1 = ¼$

$A_1 = A_2 = 0$ and $Z = 25$

Note for the Big M method

- In the final tableau, if one or more artificial variables (A_1, A_2, …) still basic and has a nonzero value, then the problem has an infeasible solution.
- All other notes are still valid in the Big M method.

Regular Simplex Method Using of Artificial Variables

3. The Tableau:-

Max: z = $2x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2 - MA_1$

Sub. To:

$$2x_1 + x_2 + x_3 + s_1 = 2$$
$$3x_1 + 4x_2 + 2x_3 - s_2 + A_1 = 8$$
$$x_1, x_2, x_3, s_1, s_2, A_1 \geq 0$$

Note: If A_1 becomes a departing variable, remove it from the whole tableau

WC

		x_1 2	x_2 2	x_3 4	s_1 0	s_2 0	A_1 -M	RHS	Ratio
s_1	0	2	1	1	1	0	0	2	2
A_1	-M	3	4*	2	0	-1	1	8	2
		-3M-2	-4M-2	-2M-4	0	M	0		

WR

Regular Simplex Method Using of Artificial Variables

5- Improving the optimality:-

		x_1	x_2	x_3	s_1	s_2	A_1	RHS	Ratio
		2	2	4	0	0	-M		
s_1	0	2	1	1	1	0	0	2	2
A_1	-M	3	4*	2	0	-1	1	8	2
		-3M-2	-4M-2	-2M-4	0	M	0		

Multiply by 1/4

+

		x_1	x_2	x_3	s_1	s_2	RHS	Ratio
		2	2	4	0	0		
s_1	0	5/4	0	1/2*	1	1/4	0	0
x_2	2	3/4	1	1/2	0	-1/4	2	4
		-1/2	0	-3	0	-1/2		

Multiply by -1

Regular Simplex Method
Using of Artificial Variables

5- Improving the optimality:-

Multiply by 2

		x_1 2	x_2 2	x_3 4	s_1 0	s_2 0	RHS	Ratio
s_1	0	5/4	0	1/2*	1	1/4	0	0
x_2	2	3/4	1	1/2	0	-1/4	2	4
		-1/2	0	-3	0	-1/2		

+

Multiply by -1/2

		x_1 2	x_2 2	x_3 4	s_1 0	s_2 0	RHS
x_3	4	5/2	0	1	2	1/2	0
x_2	2	-1/2	1	0	-1	-1/2	2
		7	0	0	6	1	

Regular Simplex Method
Using of Artificial Variables

- The optimal solution is at :

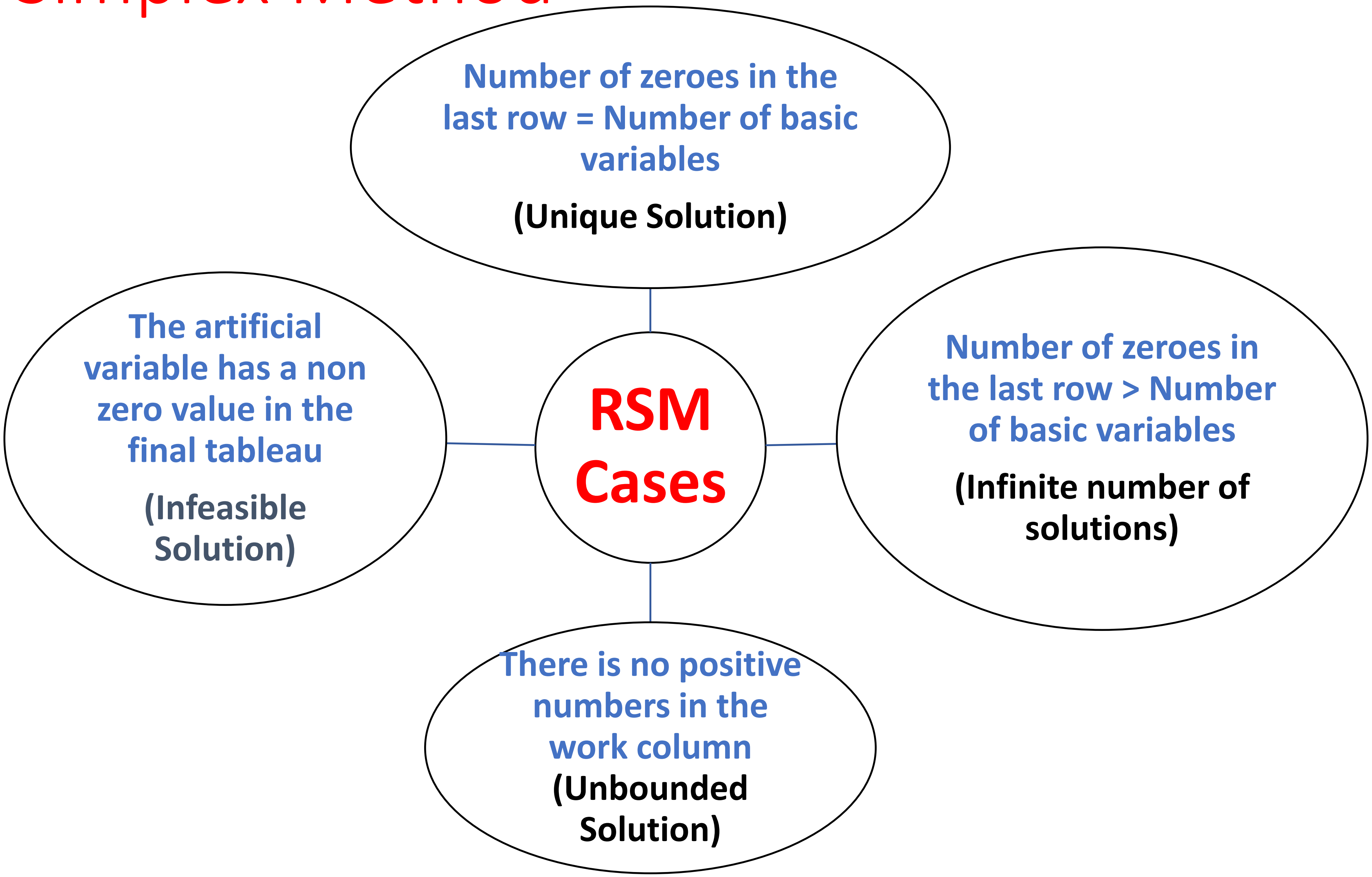
x_1 = 0, x_2 = 2, x_3 = 0, s_1 = 0, s_2 = 0, A_1= 0

Z* = 2 (0) + 2(2)+ 4(0) +0+0+0 = 4

Notes on the Simplex tableau

1. In any Simplex tableau, the intersection of any basic variable with itself is always one and the rest of the column is zeroes.
2. In any simplex tableau, the objective function row (Z row) is always in terms of the nonbasic variables. This means that under any basic variable (in any tableau) there is a zero in the last row. For the non basic there is no condition (it can take any value in this row).
3. If there is a tie (more than one variables have the same most negative or positive) in determining the entering variable, choose any variable to be the entering one.
6. If there is a tie in determining the leaving variable, choose any one to be the leaving variable. In this case a zero will appear in RHS column; therefore, a "cycle" will occur, this means that the value of the objective function will be the same for several iterations.

Regular Simplex Method



Definitions

- A basic solution is an augmented corner point solution.
- A basic solution has the following properties:

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints. Therefore, the number of nonbasic variables equals the total number of variables minus the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as simultaneous solution of the system of equations (functional constraints in augmented form). The set of basic variables are called "basis"
5. If the basic variables satisfy the nonnegativity constraints, the basic solution is a Basic Feasible (BF) solution.