

Operations Research

Lecture 4

Regular Simplex Method:
Standard Form


Outlines

- Transforming the LP problem to standard form
- Basic solutions of standard LP problem
- Regular Simplex Method

Why Standard Form?

- The simplex method had proven to be the most powerful (practical) solver of LP problems
- The implementation of simplex method requires the LP problem in standard form

What is Standard Form?

Maximize (Minimize): $Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$  Objective Function

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

.

.

.

.


$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Non-negative
Right-hand side
Constants 

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

$$b_1 \geq 0, b_2 \geq 0, \dots, b_m \geq 0$$

 Constraint Set

 Non-negative Variables Constraint

Standard Program in Matrix Form

$$\begin{array}{ll} \text{Maximize} & \mathbf{Z} = \mathbf{c}\mathbf{x} \\ \text{(Minimize):} & \\ \text{Subject to:} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{b} \geq \mathbf{0} \end{array}$$

Linear Programming - Standard Form

Maximize (Minimize): $Z = \mathbf{c}\mathbf{x}$

Subject to: $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\mathbf{x} \geq 0$$

$$\mathbf{b} \geq 0$$

Cost Vector

where,

$$\mathbf{c} = [c_1 \ c_2 \ \dots \ c_n]$$

Coefficient Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

RHS

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ b_m \end{bmatrix}$$

Var. Vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix}$$

Steps of Standard Form

- **Step 1:** *Non-negativity step*
 - RHS of all constraints must be ≥ 0
 - All variables must be Non-negative
- **Step 2:** transform inequality constraints to Equality constraints (introducing Slack and surplus variables)
- **Step 3:** Introduce Artificial Variables

Big M Method:

Form the Modified Problem

- If any problem constraints have negative constants on the right side, multiply both sides by -1 to obtain a constraint with a nonnegative constant. Remember to reverse the direction of the inequality if the constraint is an inequality.
- Introduce a *slack variable* for each constraint of the form \leq .
- Introduce a **surplus variable** and an **artificial variable** in each \geq constraint.
- Introduce an **artificial variable** in each $=$ constraint.
- For each artificial variable a , add $-Ma$ (or $+Ma$) to the objective function. Use the same constant M for all artificial variables.

Example

Write the following linear program in standard form

Maximize

$$Z = 2x_1 + x_2 - 3x_3$$

subject to

$$x_1 - x_2 - x_3 \geq -10$$

$$-x_1 + x_2 + 2x_3 \geq 2$$

$$x_1 + x_2 - 3x_3 = 20$$

$$x_1, \geq 0, x_2 \leq 0 \text{ and } x_3 \text{ unrestricted}$$

Example(continued)

Step 1: If any problem constraints have negative constants on the right side, multiply both sides by -1 to obtain a constraint with a nonnegative constant. Remember to reverse the direction of the inequality if the constraint is an inequality.

$$-x_1 + x_2 + x_3 \leq 10$$

Maximize

$$Z = 2x_1 + x_2 - 3x_3$$

subject to

$$x_1 - x_2 - x_3 \geq -10$$

$$-x_1 + x_2 + 2x_3 \geq 2$$

$$x_1 + x_2 - 3x_3 = 20$$

$$x_1, \geq 0, x_2 \leq 0 \text{ and } x_3 \text{ unrestricted}$$

Example(continued)

All variables must be Non-negative

Case 1:

Negative variables

replace with one non-negative variables

$$x_2 \leq 0$$

Let $x_2 = -x_4$ such that $x_4 \geq 0$

Maximize

$$Z = 2x_1 - x_4 - 3x_3$$

subject to

$$-x_1 - x_4 + x_3 \leq 10$$

$$-x_1 - x_4 + 2x_3 \geq 2$$

$$x_1 - x_4 - 3x_3 = 20$$

$x_1, \geq 0, x_4 \geq 0$ and x_3 unrestricted

Maximize

$$Z = 2x_1 + x_2 - 3x_3$$

subject to

$$-x_1 + x_2 + x_3 \leq 10$$

$$-x_1 + x_2 + 2x_3 \geq 2$$

$$x_1 + x_2 - 3x_3 = 20$$

$x_1, \geq 0, x_2 \leq 0$ and x_3 unrestricted

Example(continued)

All variables must be Non-negative

Case 2: Unrestricted variables:
replace with 2 non-negative
variables

x_3 unrestricted

Let $x_3 = x_5 - x_6$

such that x_5 and $x_6 \geq 0$

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6)$$

subject to

$$-x_1 - x_4 + x_5 - x_6 \leq 10$$

$$-x_1 - x_4 + 2(x_5 - x_6) \geq 2$$

$$x_1 - x_4 - 3(x_5 - x_6) = 20$$

$x_1, \geq 0, x_4 \geq 0$ and x_5 and $x_6 \geq 0$

Maximize

$$Z = 2x_1 - x_4 - 3x_3$$

subject to

$$-x_1 - x_4 + x_3 \leq 10$$

$$-x_1 - x_4 + 2x_3 \geq 2$$

$$x_1 - x_4 - 3x_3 = 20$$

$x_1, \geq 0, x_4 \geq 0$ and x_3 unrestricted

Example (continued)

Introduce a **slack variable** for each constraint of the form \leq

$$-x_1 - x_4 + x_5 - x_6 + \mathbf{S}_1 = 10.$$

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6) + \mathbf{0S}_1$$

subject to

$$-x_1 - x_4 + x_5 - x_6 + \mathbf{S}_1 = 10.$$

$$-x_1 - x_4 + 2(x_5 - x_6) \geq 2$$

$$x_1 - x_4 - 3(x_5 - x_6) = 20$$

$$x_1, \geq 0, x_4 \geq 0 \text{ and } x_5 \text{ and } x_6 \geq 0$$

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6)$$

subject to

$$-x_1 - x_4 + x_5 - x_6 \leq 10$$

$$-x_1 - x_4 + 2(x_5 - x_6) \geq 2$$

$$x_1 - x_4 - 3(x_5 - x_6) = 20$$

$$x_1, \geq 0, x_4 \geq 0 \text{ and } x_5 \text{ and } x_6 \geq 0$$

Example (continued)

- Introduce a **surplus variable** and an **artificial variable** in each \geq constraint.

$$-x_1 - x_4 + 2(x_5 - x_6) - S_2 = 2$$

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6) + 0S_1$$

subject to

$$-x_1 - x_4 + x_5 - x_6 + S_1 = 10.$$

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6) + 0S_1 + 0S_2$$

subject to

$$-x_1 - x_4 + x_5 - x_6 + S_1 = 10.$$

$$-x_1 - x_4 + 2(x_5 - x_6) - S_2 = 22$$

$$x_1 - x_4 - 3(x_5 - x_6) = 20$$

$$x_1, \geq 0, x_4 \geq 0 \text{ and } x_5 \text{ and } x_6 \geq 0$$

$$-x_1 - x_4 + 2(x_5 - x_6) \geq 2$$

$$x_1 - x_4 - 3(x_5 - x_6) = 20$$

$$x_1, \geq 0, x_4 \geq 0 \text{ and } x_5 \text{ and } x_6 \geq 0$$

Example(continued)

Introduce an **Artificial variable**

An artificial variable is a variable introduced into each equation that has a surplus variable or in = constraint.

For each artificial variable A ,

add $-MA$ to the objective function in **Maximize** Problem.

add $+MA$ to the objective function in **Minimize** Problem

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6) + 0S_1 + 0S_2 \\ - MA_1 - MA_2$$

subject to

$$-x_1 - x_4 + x_5 - x_6 + S_1 = 10.$$

$$-x_1 - x_4 + 2(x_5 - x_6) - S_2 + A_1 = 2$$

$$x_1 - x_4 - 3(x_5 - x_6) + A_2 = 20$$

$$x_1, \geq 0, x_4 \geq 0 \text{ and } x_5 \text{ and } x_6 \geq 0$$

Artificial Variables

In order to use the simplex method on problems with mixed constraints, we turn to a variable called an **artificial variable**. *This variable has no physical meaning in the original problem*

Where??? An artificial variable is a variable introduced into each equation that has a surplus variable or equal constraints.

Why?? is introduced for the purpose of obtaining a basic feasible solution so that we can apply the simplex method. To ensure that we consider only basic feasible solutions, an artificial variable is required to satisfy the nonnegative constraint.

Basic Variables???

$$s_1 \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A_1 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A_2 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ But } s_2 \rightarrow \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \text{ can't be Basic}$$

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6) + 0s_1 + 0s_2$$

$$-MA_1 - MA_2$$

subject to

$$-x_1 - x_4 + x_5 - x_6 + s_1 = 10.$$

$$-x_1 - x_4 + 2(x_5 - x_6) - s_2 + A_1 = 2$$

$$x_1 - x_4 - 3(x_5 - x_6) + A_2 = 20$$

$$x_1, \geq 0, x_4 \geq 0 \text{ and } x_5 \text{ and } x_6 \geq 0$$

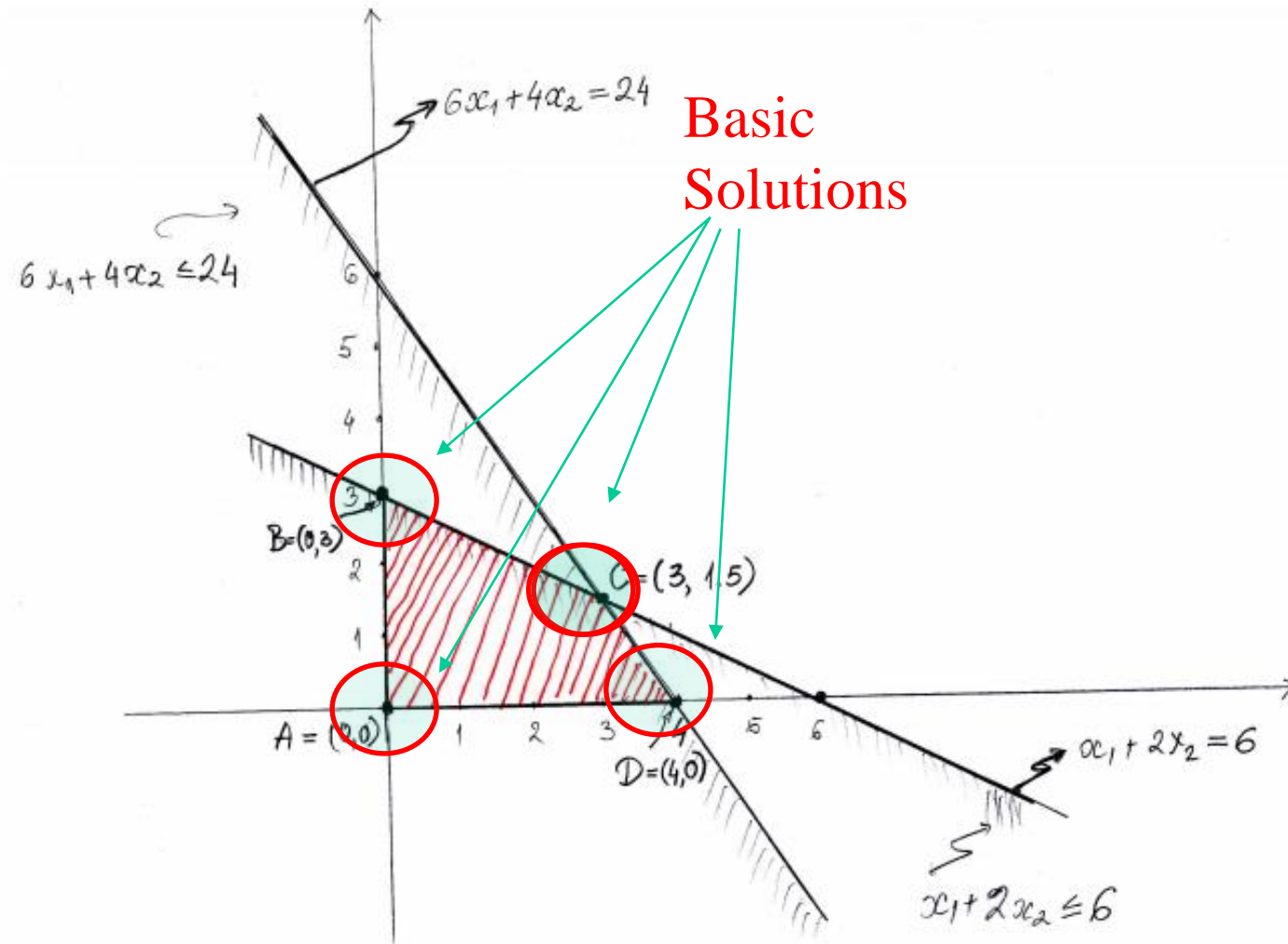
Example (continued)

Returning to our example, we introduce an artificial variable a_1 into the equation involving the surplus variable s_2 :

$$x_1 + x_2 - s_2 + A_1 = 2$$

To prevent an artificial variable from becoming part of an optimal solution to the original problem, a very large “penalty” is introduced into the objective function. This penalty is created by choosing a positive constant M so large that the artificial variable is forced to be 0 in any final optimal solution of the original problem.

Basic solutions are the corner points



What are the basic solutions?

- For a problem in the **standard form** a **basic solution** is a point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ that has at least $n - m$ coordinates equal to 0, and satisfies all the equality constraints of the problem

$$a_{11}\bar{x}_1 + a_{12}\bar{x}_2 + \dots + a_{1n}\bar{x}_n = b_1$$

$$a_{21}\bar{x}_1 + a_{22}\bar{x}_2 + \dots + a_{2n}\bar{x}_n = b_2$$

⋮

$$a_{m1}\bar{x}_1 + a_{m2}\bar{x}_2 + \dots + a_{mn}\bar{x}_n = b_m$$

- If the point \bar{x} has all components nonnegative, i.e., $\bar{x}_i \geq 0$ for all i , then \bar{x} is a **basic feasible solution**
- Otherwise, (i.e., if $\bar{x}_j < 0$ for some index j), \bar{x} is **basic infeasible solution**

How to find the basic solutions algebraically

- If the problem is not in standard form, bring it to the standard form
- Basic solutions are determined from the standard form as follows:
- Select $n - m$ out of n nonnegative inequalities (coordinate indices) i , $x_i \geq 0$, $i = 1, \dots, m$ and set them to zero

$$x_j = 0 \quad \text{for a total of } n - m \text{ indices } j \quad (\text{nonbasic variables})$$

- Substitute these zero values in the equalities:
we have m unknown variables and m equalities
- Solve this $m \times m$ system of equations: we obtain values for the remaining m variables (**basic variables**)
- If these m (basic) variables are nonnegative, we have a basic feasible solution; otherwise, a basic infeasible solution

Initial Basic Feasible Solution

Maximize

$$Z = 2x_1 - x_4 - 3(x_5 - x_6) + 0S_1 + 0S_2 - MA_1 - MA_2$$

subject to

$$-x_1 - x_4 + x_5 - x_6 + S_1 = 10.$$

$$-x_1 - x_4 + 2(x_5 - x_6) - S_2 + A_1 = 2$$

$$x_1 - x_4 - 3(x_5 - x_6) + A_2 = 20$$

$$x_1, \geq 0, x_4 \geq 0 \text{ and } x_5 \text{ and } x_6 \geq 0$$

The initial Basic Feasible solution is

$$S_1 = 10.$$

$$A_1 = 2$$

$$A_2 = 20$$

3 Basic Variables \rightarrow its number = m

(Number of constraints = 3)

$$= S_2 = 0$$

x_1, x_4, x_5 and $x_6 = 0$ Non basic Variables \rightarrow its number = n

$- m$ (Number of variables - Number of constraints) = $8 - 3 = 5$

Original LP formulation

$$\begin{aligned} &\text{maximize} && z = 5x_1 + 4x_2 \\ &\text{subject to} && 6x_1 + 4x_2 \leq 24 \\ & && x_1 + 2x_2 \leq 6 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

Standard LP form

$$\begin{aligned} &\text{maximize} && z = 5x_1 + 4x_2 \\ &\text{subject to} && 6x_1 + 4x_2 + x_3 = 24 \\ & && x_1 + 2x_2 + x_4 = 6 \\ & && x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

- We have $m = 2$ and $n = 4$
Thus, when determining the basic solutions, we set 2 indices to zero
- Suppose we choose indices $\{1, 2\}$ and set $x_1 = 0$ and $x_2 = 0$
- Substituting these in the equations yields: $x_3 = 24$ and $x_4 = 6$
- Corresponding basic solution is $x = (0, 0, 24, 6)$ and it is a basic feasible solution
- In this solution, x_1 and x_2 are nonbasic variables, while x_3 and x_4 are basic variables

Simplex Method

- *Simplex*: a linear-programming algorithm that can solve problems having more than two decision variables.

Simplex Algorithm

- The simplex technique involves **generating a series of solutions in tabular form**, called tableaus. By inspecting the bottom row of each tableau, one can immediately tell if it represents the optimal solution. **Each tableau corresponds to a corner point of the feasible solution space.** The first tableau corresponds to **the origin**. Subsequent tableaus are developed by shifting to an **adjacent corner point** in the direction that yields the highest (smallest) rate of profit (cost). This process continues as long as a positive (negative) rate of profit (cost) exists.