Operations Research

Lecture 2:
Linear Programming: Mathematical Models

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formulating linear programm

The steps for formulating the linear programming are:

1. Identify the unknown decision variables to be determined and assign symbols to them.
2. Identify the objective or aim and represent it also as a linear function of decision variables.
3. Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.

Construct linear programming model for the following problems:

Example 1

A retail store stocks two types of shirts A and B. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type A and a maximum of 300 shirts of type B. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type A shirt fetches a profit of Rs. 2 per unit and type B a profit of Rs. 5 per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

- **Step 1: Decision Var**

Let 'x_1': the store stock units of A

and 'x_2' units of B.

- **Step 2: Objective function**

As the profit contribution of A and B are Rs.2/- and Rs.5/- respectively,

objective function is: Maximize $Z = 2x_1 + 5x_2$

subjected to condition (s.t.) Structural constraints are, stores can sell 400 units of shirt A and 300 units of shirt B and the storage capacity of both put together is 600 units. Hence the structural constraints are:

$x_1 \leq 400$

$x_2 \leq 300$

for sales capacity and $x_1 + x_2 \leq 600$

Hence the model is:

Maximize: $Z = 2x_1 + 5x_2$

Subject to: $1x_1 + 0x_2 \leq 400$

$0x_1 + 1x_2 \leq 300$

$x_1 + 1x_2 \leq 600$

x_1 and x_2 are ≥ 0

Example 2

- patient consult a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin A and vitamin D. Doctor advises him to consume vitamin A and D regularly for a period of time so that he can regain his health. Doctor prescribes tonic X and tonic Y, which are having vitamin A, and D in certain proportion. Also advises the patient to consume at least 40 units of vitamin A and 50 units of vitamin D Daily. The cost of tonics X and Y and the proportion of vitamin A and D that present in X and Y are given in the table below. ***Formulate l.p. to minimize the cost of tonics.***

Vitamins	*Tonics*		*Daily requirement in units.*
	X	*Y*	
A	2	4	40
D	3	2	50
Cost in Rs. per unit.	5	3	

Example 2 cont.

- Solution: Let x be the units of X that the patient buy and y units of Y that that the patient buy .

- Objective function:

  Minimize $Z = 5x + 3y$

  s.t. $2x + 4y \geq 40$

  $3x + 2y \geq 50$ and

  Both x and y are ≥ 0.

Example 3: A company manufactures two products, X and Y by using three machines A, B, and C. Machine A has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines B and C during the coming week is 24 hours and 35 hours respectively. One unit of product X requires one hour of Machine A, 3 hours of machine B and 10 hours of machine C. Similarly one unit of product Y requires 1 hour, 8 hour and 7 hours of machine A, B and C respectively. When one unit of X is sold in the market, it yields a profit of Rs. 5 per product and that of Y is Rs. 7 per unit. Solve the problem by using graphical method to find the optimal product mix. The details given in the problem is given in the table below:

Machines	*Products (Time required in hours).*		*Available capacity in hours.*
	X	*Y*	
A	1	1	4
B	3	8	24
C	10	7	35
Profit per unit in Rs.	5	7	

Example 3 cont.

- Let the company manufactures x units of X and y units of Y, and then the L.P. model is:

Maximize $Z = 5x + 7y$

- Subject to:

$$1x + 1y \leq 4$$

$$3x + 8y \leq 24$$

$$10x + 7y \leq 35$$

Both x and y are ≥ 0.

Example 3 cont.

- As we cannot draw graph for inequalities, let us consider them as equations.
- Maximise $Z = 5x + 7y$
- s.t. $1x + 1y = 4$
- $3x + 8y = 24$
- $10x + 7y = 35$
- and both x and y are ≥ 0

x + y = 4

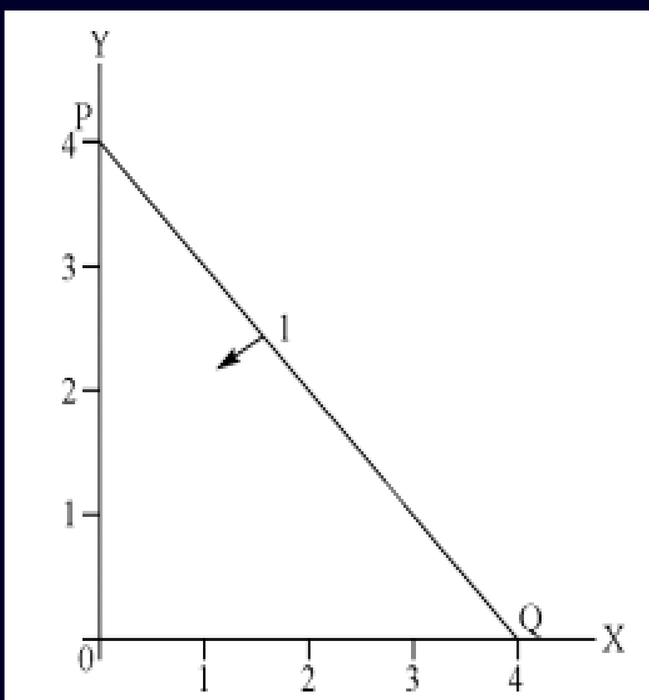
Let us take machine A. and find the boundary conditions. If x = 0, machine A can manufacture 4 units of y
Similarly, if y = 0, machine A can manufacture 4 units of x.

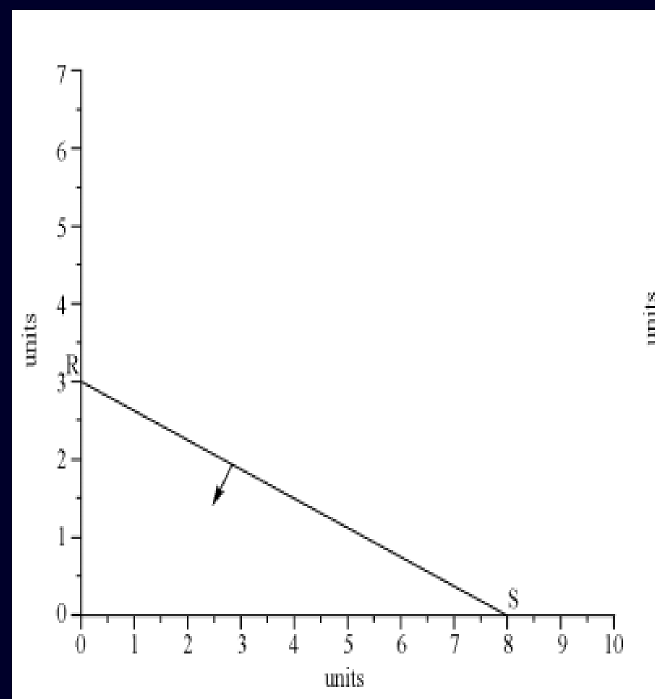
3x + 8y = 24

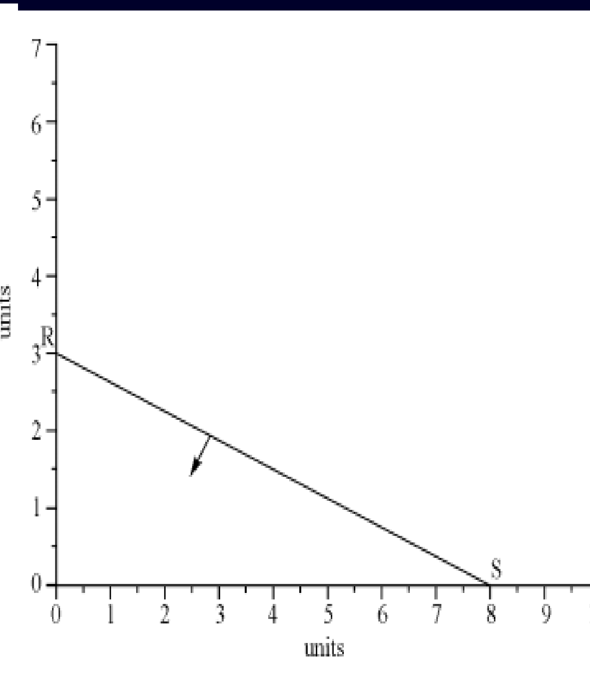
Machine B
When x = 0 , y = 3
and when y = 0 x = 8

10x + 7y = 35

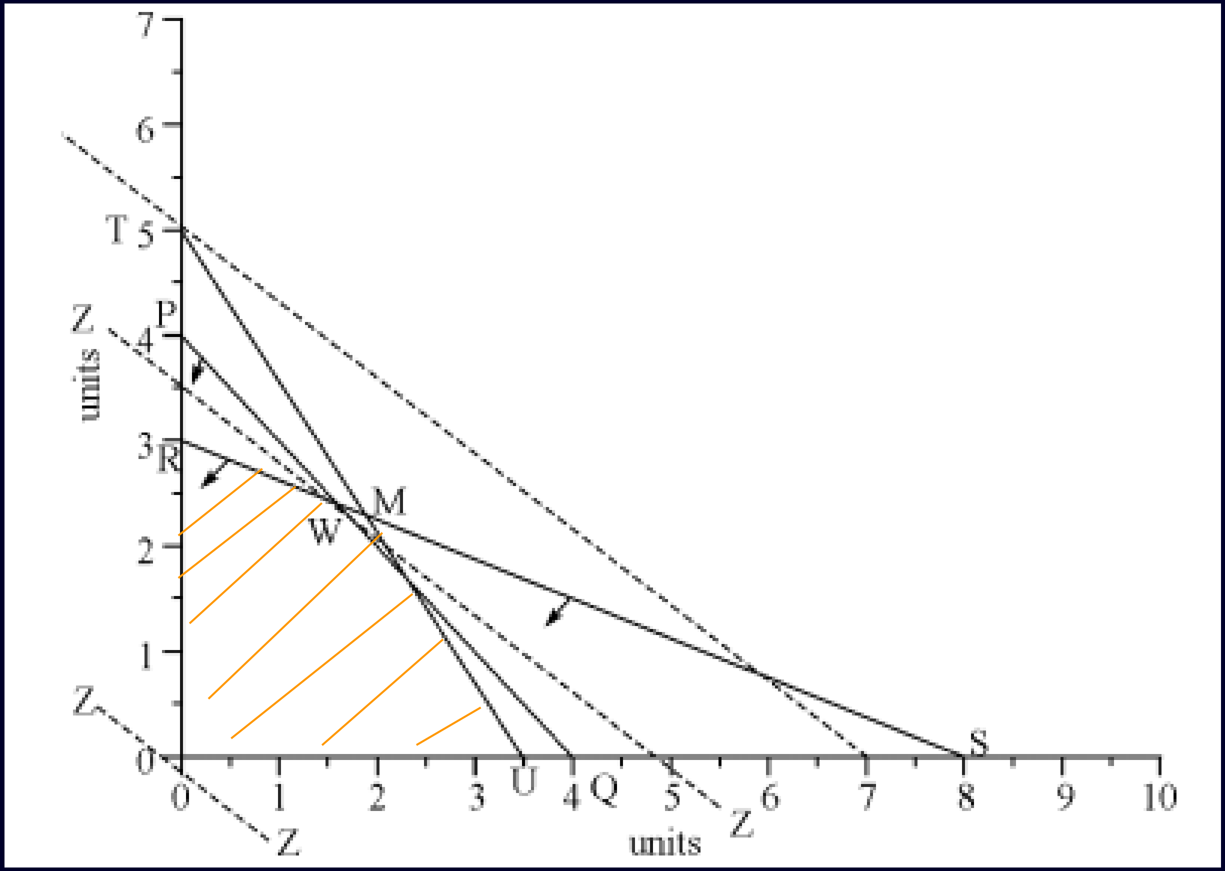
Machine C When x = 0, y = 3.5 and when y = 0, x = 5.





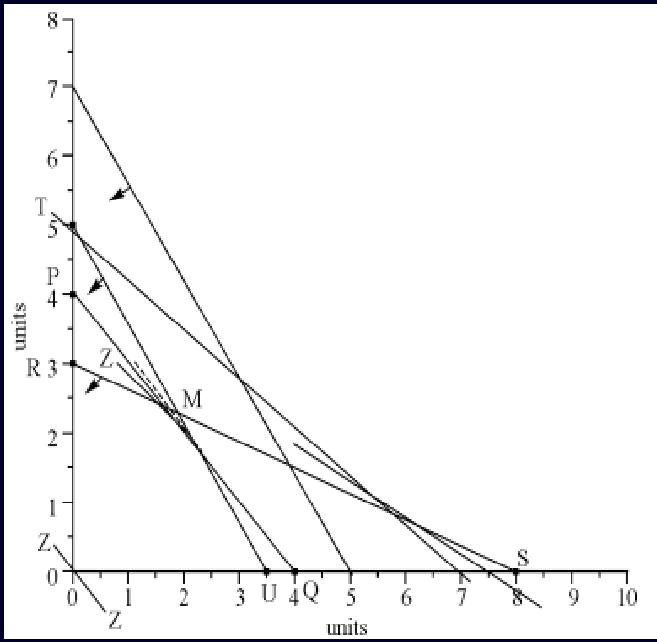


Graphical Solution to Example 3



- **Method 1.** Here we find the co-ordinates of corners of the closed polygon **ROUVW** and substitute the values in the objective function.
- In maximization problem, we select the co-ordinates giving maximum value.
- And in minimisaton problem, we select the co-ordinates, which gives minimum value.
- In the problem the co-ordinates of the corners are: R = (0, 3.5), O = (0,0), U = (3.5,0), V = (2.5, 1.5) and W = (1.6,2.4).
- Substituting these values in objective function:
- Z(0,3.5) = 5 × 0 + 7 × 3.5 = Rs. **24.50**, at point R
- Z (0,0) = 5 × 0 + 7 × 0 = Rs. **00.00**, at point O
- Z(3.5,0) = 5 × 3.5 + 7 × 0 = Rs. **17.5** at point U
- Z (2.5, 1.5) = 5 × 2.5 + 7 × 1.5 = Rs. **23.00** at point V
- **Z (1.6, 2.4) = 5 × 1.6 + 7 × 2.4 = Rs. 24.80 at point W**
- Hence the optimal solution for the problem is company has to manufacture 1.6 units of product X and 2.4 units of product Y, so that it can earn a maximum profit of Rs. 24.80 in the planning period.

Method 2. profit Line Method: profit line, a line on the graph drawn as per the objective function, assuming certain profit. On this line any point showing the values of x and y will yield same profit. For example in the given problem, the objective function is Maximize Z = 5x + 7y. If we assume a profit of Rs. 35, to get Rs. 35, the company has to manufacture either 7 units of X or 5 units of Y.



Hence, we draw line Z (preferably dotted line) for 5x + 7y = 35. Then draw parallel line to this line Z at origin. The line at origin indicates zero rupees profit. No company will be willing to earn zero rupees profit. Hence slowly move this line away from origin. Each movement shows a certain profit, which is greater than Rs.0.00. While moving it touches corners of the polygon showing certain higher profit. Finally, it touches the farthermost corner covering all the area of the closed polygon. This point where the line passes (farthermost point) is the OPTIMAL SOLUTION of the problem. In the figure 2.6. the line ZZ passing through point W covers the entire area of the polygon, hence it is the point that yields highest profit. Now point W has co-ordinates (1.6, 2.4). Now Optimal profit Z = 5 × 1.6 + 7 × 2.4 = Rs. 24.80.

Points to be Noted:

If the profit line passes through single point, it means to say that the problem has unique solution.

(i) If the profit line coincides any one line of the polygon, then all the points on the line are solutions, yielding the same profit. Hence the problem has infinite solutions.

(ii) If the line do not pass through any point (in case of open polygons), then the problem does not have solution, and we say that the problem is *UNBOUND*.

Example 4: Product Mix Problem

- A company manufactures three products namely X, Y and Z. Each of the product require processing on three machines, Turning, Milling and Grinding. Product X requires 10 hours of turning, 5 hours of milling and 1 hour of grinding. Product Y requires 5 hours of turning, 10 hours of milling and 1 hour of grinding, and Product Z requires 2 hours of turning, 4 hours of milling and 2 hours of grinding. In the coming planning period, 2700 hours of turning, 2200 hours of milling and 500 hours of grinding are available. The profit contribution of X, Y and Z are Rs. 10, Rs.15 and Rs. 20 per unit respectively. Find the optimal product mix to maximize the profit.

Solution: The given data can be written in a table.

Machine	*Product* *Time required in hours per unit*			*Available hours*
	X	*Y*	Z	
Turning.	10	5	2	2,700
Milling	5	10	4	2,200
Grinding.	1	1	2	500
Profit contribution in Rs. per unit.	10	15	20	

Example 4 cont.

- Let the company manufacture x units of X, y units of Y and z units of Z Inequalities: Equations:

- **Maximize: $Z = 10x + 15y + 20z$**

Subject to: $10x + 5y + 2z \leq 2700$

$5x + 10y + 4z \leq 2{,}200$

$1x + 1y + 2z \leq 500$

All x, y and z are ≥ 0

Example 5 : Manpower Problem

- A truck company requires the following number of drivers for its trucks during 24 hours: According to the shift schedule, a driver works eight consecutive hours, starting at the beginning of one of the six periods. Determine a daily driver worksheet which satisfies the requirements with the least number of drivers. (Formulate the mathematical program only)

Period	1	2	3	4	5	6
time	00-04	04-08	08-12	12-16	16-20	20-24
Min required number	5	10	20	12	22	8

Example 5 : Manpower Problem cont.

Period	1	2	3	4	5	6
time	00-04	04-08	08-12	12-16	16-20	20-24
Min required number	5	10	20	12	22	8

Let x_i *is the number of drivers begining their shift at period i*

$$x_1 + x_6 \geq 5$$
$$x_1 + x_2 \geq 10$$
$$x_2 + x_3 \geq 20$$
$$x_3 + x_4 \geq 12$$
$$x_4 + x_5 \geq 22$$
$$x_5 + x_6 \geq 8$$

With all variables ≥ 0 and integer

Example 6: Transportation Problem

- Four factories, A, B, C and D produce sugar and the capacity of each factory is given below: Factory A produces 10 tons of sugar and B produces 8 tons of sugar, C produces 5 tons of sugar and that of D is 6 tons of sugar. The sugar has demand in three markets X, Y and Z. The demand of market X is 7 tons, that of market Y is 12 tons and the demand of market Z is 4 tons. The following matrix gives the transportation cost of 1 ton of sugar from each factory to the destinations.
- **Find the optimal solution for least transportation cost.**

Factories.	Cost in Rs. per ton (× 100) Markets.			Availability in tons.
	X	Y	Z	
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\Sigma\ b = 29$, $\Sigma\ d = 23$

x_{ij}: *the number of units to be transported from f_i to Market j*

Minimize: Z= $4x_{11} + 3x_{12} + 2x_{13} \ldots + 6x_{31} + 4x_{32} + 3x_{33} + 3x_{41} + 5x_{42} + 4x_{43}$

Subject to:

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &\le 10 \\ x_{21} + x_{22} + x_{23} &\le 8 \\ x_{31} + x_{32} + x_{33} &\le 5 \\ x_{41} + x_{42} + x_{43} &\le 6 \end{aligned}\right\} \text{Supply constraints}$$

(because the sum must be less than or equal to the available capacity)

Transportation Problem cont.

$$\left.\begin{aligned} x_{11}+x_{21}+x_{31}+x_{41} &\geq 7 \\ x_{12}+x_{22}+x_{32}+x_{42} &\geq 12 \\ x_{13}+x_{23}+x_{33}+x_{43} &\geq 4 \end{aligned}\right\}$$

Demand constraints

$x_{ij} \geq 0$ where i=1,2,3,4 and j=1,2,3

(This is because we cannot supply negative elements)

If the **total Supply = the Total Demand** then we can rewrite all the constraints as <u>equal</u> constraints

Example 7: Assignment Model

- There are 3 jobs A, B, and C and three machines X, Y, and Z. All the jobs can be processed on all machines. The time required for processing job on a machine is given below in the form of matrix. Make allocation to minimize the total processing time.

Machines (time in hours)			
Jobs	*X*	*Y*	*Z*
A	11	16	21
B	20	13	17
C	13	15	12

Example 7: Assignment Model

- A legal firm has accepted five new cases, each of which can be handled by any one of its five junior partners. Due to difference in experience and expertise, however, the junior partners would spend varying amounts of time on the cases. A senior partner has estimated the time required in hours as shown below

Determine the optimal assignment of cases to lawyers such that each junior partner receives a different case. And the total hours expected by the firm is minimized. Formulate the corresponding mathematical program.

	Case1	Case2	Case3	Case4	Case5
Lawyer1	145	122	130	95	115
Lawyer2	80	63	85	48	78
Lawyer3	121	107	93	69	95
Lawyer4	118	83	116	80	105
Lawyer5	97	75	120	80	111

	Case1	Case2	Case3	Case4	Case5
Lawyer1	145	122	130	95	115
Lawyer2	80	63	85	48	78
Lawyer3	121	107	93	69	95
Lawyer4	118	83	116	80	105
Lawyer5	97	75	120	80	111

x_{ij}: *the number of times that Lawyeri assigned to case j*

Minimize: Z=$145x_{11} + 122x_{12} + 130x_{13} \ldots + 121x_{31} + 107x_{32} + 93x_{33} + 118x_{41} + 83x_{42} + 116x_{43} + 80x_{44} + 105x_{45} \ldots + 111x_{55}$

Subject to:

$$\left.\begin{aligned} \boldsymbol{x_{11} + x_{12} + x_{13} + x_{14} + x_{15}} &= \mathbf{1} \\ \boldsymbol{x_{21} + x_{22} + x_{23} + x_{24} + x_{25}} &= \mathbf{1} \\ \boldsymbol{x_{31} + x_{32} + x_{33} + x_{34} + x_{35}} &= \mathbf{1} \\ \boldsymbol{x_{41} + x_{42} + x_{43} + x_{44} + x_{45}} &= \mathbf{1} \\ \boldsymbol{x_{51} + x_{52} + x_{53} + x_{54} + x_{55}} &= \mathbf{1} \end{aligned}\right\}$$

Each lawyer is assigned to one case

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} + x_{41} + x_{51} &= 1 \\ x_{12} + x_{22} + x_{32} + x_{42} + x_{52} &= 1 \\ x_{13} + x_{23} + x_{33} + x_{43} + x_{53} &= 1 \\ x_{14} + x_{24} + x_{34} + x_{44} + x_{54} &= 1 \\ x_{15} + x_{25} + x_{35} + x_{45} + x_{55} &= 1 \end{aligned}\right\}$$

Each case is assigned to one lawyer

with all variables non-negative and integer

What if???

Number of Cases is Less than the number of Lawyers (3 cases and 5 lawyers)

3 cases and 5 Lawyers

$$\mathbf{x_{11} + x_{12} + x_{13} \leq 1}$$
$$\mathbf{x_{21} + x_{22} + x_{23} \leq 1}$$
$$\mathbf{x_{31} + x_{32} + x_{33} \leq 1}$$
$$\mathbf{x_{41} + x_{42} + x_{43} \leq 1}$$
$$\mathbf{x_{51} + x_{52} + x_{53} \leq 1}$$

$$\left.\begin{array}{r} x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 1 \\ x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 1 \\ x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1 \end{array}\right\}$$

Example 8: Inspection Model

- A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1,500 pieces must be inspected in an 8-hours day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%. Wages of grade I inspector are $5 per hour while those of grade II inspector are $4 per hour. Any error made by an inspector costs $3 to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company, find the optimal assignment of inspectors that minimize the daily inspection cost (Formulate only the mathematical problem).

Example 8: Inspection Model

- Let x the number of grade I that may be assigned the job of quality control inspection
- Let y the number of grade II that may be assigned the job of quality control inspection

- The objective is to minimize the daily cost of inspection

- Two cost: wages paid by inspectors and the cost of inspector error

	Grade I	Grade II
	Check 20 pieces /h	Check 14 pieces /h
Accuracy	96%	92%
costs Error 3	0.04	0.08
Wages	5	4
available	10 inspectors	15

- The cost of grade I inspector per hour
- (5+3x0.04x20)=7.4
- The cost of grade II inspector per hour

(4+3x0.08x14)=7.36

The objective function

Z=8 (7.4x+7.36y)

Example 8: Inspection Model

- $x \leq 10$
- $y \leq 15$
- $20*8x+14*8y \geq 1500$
- With all variable non-negative and integer


Thank
You