Vector space and Subspace

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$. $2 \times \frac{1}{2} = 1$

Vectors in $\mathbb{R}^3$

$(\mathbb{R}^3, \oplus, \cdot)$ $\quad \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$, $c\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}$

$u_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $u_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, 1. $u_1 + u_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \in \mathbb{R}^3$

2. $(u_1 + u_2) = u_2 + u_1$, 3. $(u_1 + u_2) + u_3 = \begin{pmatrix} (x_1 + x_2) + x_3 \\ (y_1 + y_2) + y_3 \\ (z_1 + z_2) + z_3 \end{pmatrix} = \begin{pmatrix} x_1 + (x_2 + x_3) \\ y_1 + (y_2 + y_3) \\ z_1 + (z_2 + z_3) \end{pmatrix} = u_1 + (u_2 + u_3)$

4. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$, 5. $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $-u = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$, $u - u = 0$

6. $cu = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \in \mathbb{R}^3$, 10. $1 \cdot u = \begin{pmatrix} 1 \cdot x \\ 1 \cdot y \\ 1 \cdot z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = u$

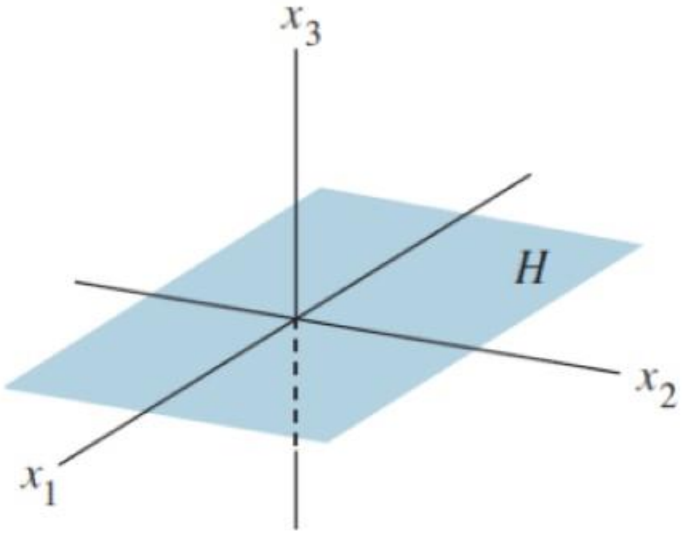
A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$



is a subset of $\mathbb{R}^3$ that "looks" and "acts" like $\mathbb{R}^2$, although it is logically distinct from $\mathbb{R}^2$. See Fig. 7. Show that H is a subspace of $\mathbb{R}^3$.

The zero vector of V is in H

$$\begin{pmatrix} s \\ t \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in H$$

$$\underline{\underline{H}} = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

$$V = \mathbb{R}^3 \ , \quad 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

H is closed under vector addition. $u_1 = \begin{pmatrix} s_1 \\ t_1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} s_2 \\ t_2 \\ 0 \end{pmatrix} \in H$; $s_1, t_1, s_2, t_2 \in \mathbb{R}$

$$u_1 + u_2 = \begin{pmatrix} s_1 + s_2 \\ t_1 + t_2 \\ 0 \end{pmatrix} \begin{matrix} \in \mathbb{R} \\ \in \mathbb{R} \end{matrix} = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} \in H$$

$$H \leq V = \mathbb{R}^3$$

H is closed under multiplication by scalars $u = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix}$, $c \in \mathbb{R}$

$$cu = \begin{pmatrix} cs \\ ct \\ 0 \end{pmatrix} \begin{matrix} \in \mathbb{R} \\ \in \mathbb{R} \end{matrix} \begin{pmatrix} s' \\ t' \\ 0 \end{pmatrix} \in H$$

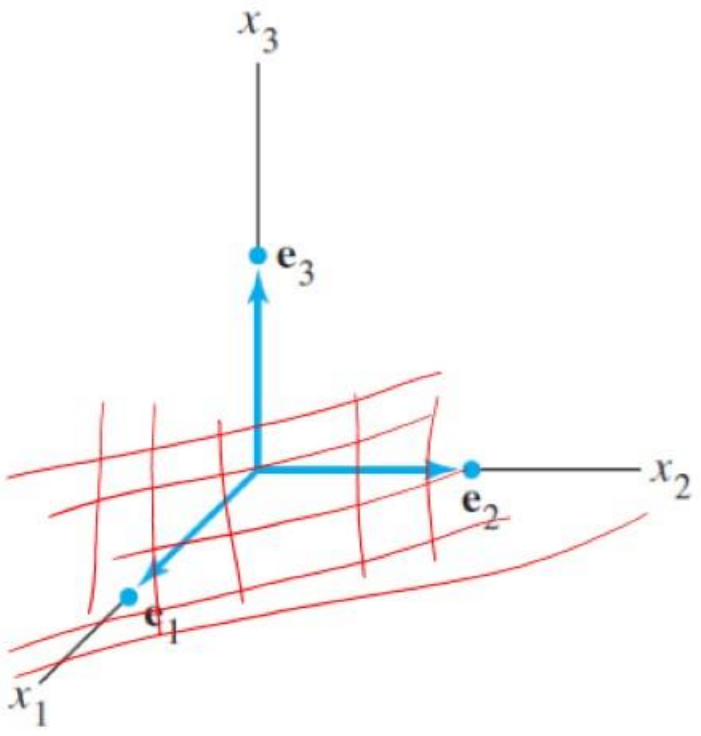
$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\} = \operatorname{span}\{e_1, e_2\} \leqslant \mathbb{R}^3$$

$$\begin{pmatrix} s \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = s e_1 + t e_2$$

$\{0\} \leqslant V$

any span $\leqslant V$

$V \leqslant V$



$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

$\times$ 1 2

Linearly independent but does not span $\mathbb{R}^3$

A basis for $\mathbb{R}^3$

$$\begin{bmatrix}1 & 2 & 4\\0 & 3 & 5\\0 & 0 & 6\end{bmatrix}$$

$\times$ Spans $\mathbb{R}^3$ but is linearly dependent

For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u} = \mathbf{0} \tag{1}$$

$$c\mathbf{0} = \mathbf{0} \tag{2}$$

$$-\mathbf{u} = (-1)\mathbf{u} \tag{3}$$